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ON THE CLASS OPERATOR OF THE SU(2) GROUP

ABSTRACT. The explicit expression for the class operator of the SU(2) group, previously obtained by the integration with ordered product technique, is rederived in a more direct way.

1. Introduction

Some years ago, Fan Hong-yi and Ren Yong [1] gave, for the class operator of the SU(2) group

\[ C(\alpha) = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \exp \left[ i \alpha (J_x \sin \theta \cos \phi + J_y \sin \theta \sin \phi + J_z \cos \theta) \right] \]

the explicit formula

\[ C(\alpha) = \frac{4\pi}{\sin \left( \frac{\alpha}{2} \right)} \frac{\sin \left( \frac{\alpha}{2} \sqrt{4J^2 + 1} \right)}{\sqrt{4J^2 + 1}}. \]  

To overcome the difficulty that the angular momentum operators do not commute with each other, they first wrote the exponential operator appearing in (1) in terms of the Schwinger boson representation [2], and then applied the integration within ordered product technique [3]. However, in spite of the simplicity of the result, the calculations are somewhat laborious. In this note we provide a more direct derivation of equation (2), based on the formula [4,5]

\[ e^{i\alpha n \cdot J} = e^{J^+} e^{2\nu J^z} e^{J^-}, \]

where

\[ n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \] (4)

\[ J^\pm = J_x \pm iJ_y \] (5)

\[ e^{i\alpha n \cdot J} = e^{J^+} e^{2\nu J^z} e^{J^-}, \]
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\[ u^\pm = \frac{i \sin \theta e^{\pm i \phi}}{\cot \frac{\phi}{2} + i \cos \theta}; \quad v = \ln \left( \cos \frac{\alpha}{2} + i \cos \theta \sin \frac{\alpha}{2} \right). \]  

(6)

As we shall see, equation (3), together with the fact that the class operator is only related to \( J^2 \), enables us to achieve the goal in a quite simple manner. This is done in Section 2. In Section 3, as a further application of equation (3), we derive a sum rule for Jacobi polynomials.

2. Proof of equation (2)

The underlying idea is to consider the matrix element \( \langle j, m | C(\alpha) | j, m \rangle \). By using (3), and

\[ (J^\pm)^l | j, m \rangle = \left[ \frac{(j + m)!}{(j - m - l)!} \right]^{1/2} | j, m \pm l \rangle \]  

(7)

a straightforward calculation gives

\[ \langle j, m | e^{i \alpha \cdot J} | j, m \rangle = e^{2m \nu} _2F_1(-j + m, j + m + 1; 1; -u^+ u^- e^{2\nu}). \]  

(8)

Therefore

\[ \langle j, m | C(\alpha) | j, m \rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta \left( \cos \frac{\alpha}{2} + i \cos \theta \sin \frac{\alpha}{2} \right)^{2m} \cdot _2F_1(-j + m, j + m + 1; 1; \sin^2 \theta \sin^2 \frac{\alpha}{2}). \]  

(9)

Since the left hand side does not depend on \( m \), we evaluate the right hand side with \( m = j \). Thus

\[ \langle j, m | C(\alpha) | j, m \rangle = \frac{4\pi}{2j + 1} \frac{\sin((2j + 1) \frac{\pi}{2})}{\sin(\frac{\pi}{2})}. \]  

and, since \( (2j + 1)^{2l} = \langle j, m | (4J^2 + 1)^l | j, m \rangle \) (\( l = 0, 1, 2, \ldots \)), equation (2) follows at once.

It is instructive to check explicitly that the integral in (9) is \( m \)-independent, as it should be. Let us write

\[ I_m = i \sin \frac{\alpha}{2} \int_0^{\pi} d\theta \sin \theta A^{2m} _2F_1(-j + m, j + m + 1; 1; \sin^2 \theta \sin^2 \frac{\alpha}{2}) \]  

(11)