The inspection paradox with random time

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When considering a delayed renewal process one may be interested in both, the renewal function and the expected length of the interarrival time that contains some fixed time $t$. In general, it is difficult to obtain explicit expressions for specific underlying distributions. Replacing $t$ by a random variable $T$ and using prior information about $T$, that is, assuming that $T$ has some continuous NBU (NWU) distribution function $G$, bounds of the quantities are derived as well as representations, if $T$ is exponentially distributed. As an implication an equation of Wald type is shown. The results can be applied to the analysis of control charts in quality control. Moreover, related bounds of a sample mean based on a random sample size are given and an elementary renewal reward theorem is stated.

Keywords: Delayed renewal process, renewal function, inspection paradox.

1. Introduction

In many practical situations, a renewal process serves as the basic statistical model. We refer to reliability theory, models in non-life insurance, queuing theory, inventory and traffic flow models.

Let $(S_n)_{n \in \mathbb{N}}$ be a delayed renewal process defined by

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N},$$

with independent, nonnegative interarrival times and distribution functions

$$X_1 \sim F_1, \quad X_i \sim F, \quad i \geq 2, \quad F_1(0) < 1, \quad F(0) < 1.$$

Then $(N(t))_{t \geq 0}$ with $N(t) = \sup \{ n \in \mathbb{N}_0 : S_n \leq t \}$ denotes the corresponding renewal counting process.
In the above examples, \( X_i \) may describe, respectively, the life length of the \( i \)th component (i.e. the time between successive failures), the waiting time between two successive claims or the size of the \( i \)th claim, the interarrival time of two customers, the demand in the \( i \)th period, and the time between successive road users.

If we consider a common renewal process, i.e. \( F_1 = F \), and the interarrival time that contains some fixed time \( t \), namely \( X_{N(t)+1} \), then the well-known inspection paradox (e.g. Ross 1983a) yields that \( X_{N(t)+1} \) is stochastically larger than \( X_1 \)

\[
P(X_{N(t)+1} > x) \geq P(X_1 > x) \quad \text{for all } x \geq 0
\]

which implies

\[
E X_{N(t)+1} \geq E X_1.
\]

The survival function of \( X_{N(t)+1} \) is given by

\[
P(X_{N(t)+1} > x) = \begin{cases} 
F(x) \left( 1 + \frac{EN(t)}{F(t)} \right), & x > t \\
F(t) + \int_0^{t-x} \frac{1}{F(t-s)} dEN(s) + F(x) \left( \frac{1}{EN(t)} - \frac{1}{EN(t-x)} \right), & x \leq t
\end{cases}
\]

Hence, explicit expressions of \( E X_{N(t)+1} \) are available for some particular distributions only.

When inspecting the underlying renewal process at time \( t \), we are usually interested in the expected length of the present interarrival time and in the expected number of renewals, claims, customers, etc. up to time \( t \). In practice, inspections may happen at random or they are supposed to be random. Thus, in order to appraise the values of the above quantities, the fixed time \( t \) has to be replaced by a random time \( T \). We suppose to have prior information about the distribution of \( T \):

\[
Let \( T \) be a nonnegative random variable with some continuous distribution function \( G \), \( G(x) < 1 \) for all \( x > 0 \), which is independent of the interarrival times \( (X_i)_{i \in \mathbb{N}} \).
\]

Obviously, the inequalities (1.1) and (1.2) remain to hold true with \( t \) replaced by \( T \). Moreover, \( G \) is assumed to be NBU (or NWU) which means that

\[
G(x + y) \leq (\geq) \ G(x) \cdot G(y) \quad \text{for all } x,y \geq 0.
\]