SINGULAR INTEGRAL EQUATIONS ON CLOSED CONTOURS

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SOMMARIO. — Vengono costruite soluzioni per equazioni integrali singolari su contorni chiusi utilizzando una generalizzazione delle relazioni di ortogonalità della teoria del trasporto. Si ottengono semplificazioni notevoli rispetto alla teoria classica, che fa uso della trasformazione di Hilbert.

1. - INTRODUCTION.

Recently [1], the authors proposed to solve singular integral equations of Cauchy type on intervals, such as commonly arise in transport theory [2], by an orthogonality method. Similar methods were introduced in transport theory long ago [3] but were restricted specifically to the transport equation. The point of Ref. 1 was that a similar method could be used for quite general equations. This approach has a number of advantages. Among them are elegance and simplicity (the Hilbert transform used in the standard method of solving singular integral equations [4, 5] need not be introduced); familiarity (the method of solution becomes closely analogous to classical techniques for solving partial differential equations); and, perhaps most important, insight (for example, the so-called endpoint conditions [6] usually introduced in a completely ad hoc manner are seen to arise naturally, as a condition that certain contour integrals exist).

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In the present work the analysis of Ref. 1 is extended to the case that the contour of integration is a closed rectifiable Jordan curve in the complex plane which we assume to be the unit circle $S^1$. Of course, endpoint conditions are no longer involved, but we find advantages in this case also. In particular, for certain problems (the so-called «non-normal» problems of Ref. 5) the gain in simplicity is quite spectacular. Also, certain problems not previously solved can be treated by a limiting procedure. These points are elaborated in more detail in Section 3.

2. - THE BASIC METHOD.

We consider the equation

\[ f(t) = \lambda(t) A(t) + P \int_{-1}^{1} \frac{\eta(v) A(v)}{v - t} \, dv, \]

where the symbol $P$ mean Cauchy principal value and the contour integration is, as mentioned previously, the unit circle $S^1$. We seek complex-valued uniformly Hölder continuous solutions $A(t)$ for $\|t\| = 1$, under the assumption that the given functions $f$, $\lambda$ and $\eta$ are uniformly Hölder continuous on $S^1$.

The more general problem

\[ g(t) = \lambda(t) B(t) + \eta_1(t) P \int_{-1}^{1} \frac{\eta_2(v) B(v)}{v - t} \, dv \]

can be reduced to Eq. (1) by the substitutions

\[ B(v) = \eta_1(v) A(v), \]
\[ g(t) = \eta_1(t) f(t), \]
\[ \eta_1(t) \eta_2(t) = \eta(t), \]

so we shall consider only Eq. (1).

In solving Eq. (1), it is necessary, as in Ref. 1, to introduce the solution $X(z)$ to a homogeneous Riemann-Hilbert problem, i.e. we seek a solution holomorphic on $\mathbb{C} \setminus S^1$ whose boundary values $X^\pm$ satisfy

\[ \frac{X^+(t)}{X^-(t)} = \frac{\lambda(t) + \pi i \eta(t)}{\lambda(t) - \pi i \eta(t)}, \quad t \in S^1, \]

with

\[ X^\pm(t) = \lim_{\varepsilon \to 0} X((1 \pm \varepsilon) t). \]