W. ZELAZKO

dell'Istituto di Matematica, Academia Polacca delle Scienze

ON CERTAIN OPEN PROBLEMS
IN TOPOLOGICAL ALGEBRAS

(Conferenza tenuta il 20 marzo 1989)


Gran parte dei problemi riportati qui o in [25] sono pubblicati per la prima volta.

A topological algebra is a topological linear space over real or complex scalars on which is defined an associative jointly continuous multiplication. That means that if \( \Phi(A) \) is a basis of neighbourhoods of the origin in a topological algebra \( A \), then for each \( U \) in \( \Phi(A) \) there is a \( V \in \Phi(A) \) such that

\[
V^2 \subset U.
\]

We denote by \( T \) the class of all topological algebras.

By a locally convex algebra we mean a topological algebra whose underlying topological linear space is locally convex. The topology of such an algebra is given by means of a family \( (\| \cdot \|) \) of seminorms, and from (1) it follows that these seminorms can be so choosen that for each index \( \alpha \) there is an index \( \beta \) such that

\[
\|xy\|_\alpha \leq \|x\|_\beta \|y\|_\beta
\]

for all elements \( x \) and \( y \) of the algebra in question. The class of all
locally convex algebras will be denoted by $LC$. In the case when in an $LC$-algebra $A$ the condition (2) can be replaced by

\[(3) \quad ||xy||_a \leq ||x||_a ||y||_a\]

for all $x, y \in A$, we call such an algebra to be multiplicatively-convex (shortly $m$-convex). The class of all $m$-convex algebras will be denoted by $MLC$. The class of all topological algebras whose underlying spaces are $F$-spaces ($= \text{completely metrizable topological linear spaces}$) will be denoted by $F$. Observe that the completeness condition is not very restrictive for the topological algebras, since, by the condition (1), the completion of a topological algebra is again such an algebra, i.e. the multiplication of a topological algebra extends to a jointly continuous multiplication in its completion. This would be not longer true if we replace the condition of joint continuity of the multiplication by the condition of separate continuity of the product. There exist separately continuous multiplications on normed spaces which do not extend to the completions of these spaces. The topological linear spaces which are algebras with separately continuous multiplication we shall call semi-topological algebras. It can be shown that a semi-topological algebra of type $F$ (i.e. its underlying topological linear space is of type $F$) must be a topological algebra (see [2] or [9], theorem 7.2). Similarly as in the topological algebra case we shall call a semi-topological algebra locally convex if its underlying space is locally convex. Further we put $B_0 := LC \cap F$, so a $B_0$-algebra, or an algebra of type $F$, is a completely metrizable locally convex algebra. Its topology can be given by means of an increasing sequence of seminorms

\[||x||_1 \leq ||x||_2 \leq \ldots\]

and the condition (2) reads now as

\[(4) \quad ||xy||_i \leq ||x||_{i+1} ||y||_{i+1}, \quad x, y \in A, \quad i = 1, 2, \ldots\]

The class of $m$-convex $B_0$-algebras will be denoted by $MB_0$, the class of Banach algebras will be denoted by $B$ and the class of finite-dimensional algebras will be denoted by $FD$. Between these classes of topological algebras we have the following relations

\[FD \subset B \subset MB_0 \subset B_0 \subset LC \subset T,\]

\[MB_0 \subset MLC \subset LC \text{ and } B_0 \subset F \subset T.\]