Abstract. We are interested in variational problems of the form
\[
\min \int W(\nabla u) \, dx,
\]
with \(W\) nonconvex. The theory of relaxation allows one to calculate the minimum value, but it does not determine a well-defined "solution" since minimizing sequences are far from unique. A natural idea for determining a solution is regularization, i.e. the addition of a higher order term such as \(\epsilon|\nabla^2 u|^2\). But what is the behavior of the regularized solution in the limit as \(\epsilon \to 0\)? Little is known in general.

Our recent work [19, 20, 21] discusses a particular problem of this type, namely \(\min_{u_x=\pm 1} \int \int u_x^2 + \epsilon|u_{yy}| \, dx \, dy\) with various boundary conditions. The present paper gives an expository overview of our methods and results.
1 Introduction

This paper provides an expository review of our recent work [19, 20, 21]. The general setting is as follows. We are concerned with nonconvex (and non-quasiconvex) variational problems of the form

$$\min_{u = u_0 \text{ at } \partial \Omega} \int_{\Omega} W(\nabla u) \, dx.$$  \hspace{1cm} (1)

Such problems arise naturally in a variety of contexts, including structural optimization and the modelling of martensitic phase transformation [3, 4, 17, 22].

Since $W$ is assumed to be neither convex nor quasiconvex, the minimum in (1) may not be achieved [9]. It is natural to be suspicious of a problem that "has no solution." The simplest way to restore existence is by regularization, i.e. by adding a term involving higher derivatives:

$$\min_{u = u_0 \text{ at } \partial \Omega} \int_{\Omega} W(\nabla u) + \epsilon^2 |\nabla \nabla u|^2 \, dx.$$  \hspace{1cm} (2)

This type of regularization is standard in the literature on coherent phase transformation, see e.g. [15]. Another kind, not much different in practice, is the inclusion of surface energy at phase interfaces, see e.g. [12].

There are two different approaches to this class of problems. The first, more traditional one is to discard (1) as having no sense, and concentrate instead on (2). To make contact with (1), one should then consider the behavior of the minimizer $u_\epsilon$ as $\epsilon$ tends to zero. The Euler-Lagrange equation is a fourth-order partial differential equation. As $\epsilon \to 0$ there will be a sequence of bifurcations, and one should follow the "principal branch," i.e. the one representing the minimum rather than a saddle point.