THE KUNZE-STEIN PHENOMENON

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SUNTO. — Si dà una nuova dimostrazione di un teorema di Kunze e Stein, che dice che, se $1 \leq p < 2$, $L^p(SL(2, R)) \ast L^q(SL(2, R))$ è contenuto in $L^q(SL(2, R))$. Questa nuova dimostrazione può essere generalizzata per provare lo stesso teorema per ogni gruppo di Lie connesso, semisemplice, col centro finito.

INTRODUCTION. - Let $G$ be a locally compact group with left Haar measure $m$; $L^p(G)$ denotes the usual Lebesgue space relative to this measure. We write $u \ast v$ for the convolution of the functions $u$ and $v$ on $G$, defined thus:

$$u \ast v(g') = \int_G d m(g) \ u(g'g) \ v(g^{-1}) \ g' \in G.$$

It is well known that $L^1(G) \ast L^2(G) \subseteq L^2(G)$ and

$$\|u \ast v\|_2 \leq \|u\|_1 \ \|v\|_2$$

for $u$ and $v$ in $L^1(G)$ and $L^2(G)$ respectively; if $G$ is compact, then $L^p(G) \subseteq L^1(G)$ for all $p$ greater than one, so in this case, $L^p(G) \ast L^q(G) \subseteq L^2(G)$. But if $G$ is abelian, or, more generally, amenable and noncompact, then $L^p(G) \ast L^q(G) \subseteq L^2(G)$ only if $p = 1$. The following result therefore comes as a surprise.

THEOREM. - Let $G$ be a connected semisimple Lie group with finite center. If $1 \leq p < 2$, then $L^p(G) \ast L^q(G) \subseteq L^2(G)$.

This theorem is in fact already known in some special cases. Many years ago, R. A. Kunze and E. M. Stein proved it for $SL(2, R)$ [4]. More recently, R. L. Lipsman [5], [6], [7] treated the groups $SO_o(n, 1)$ and $SL(n, C)$. Then Stein [8] tackled the complex classical groups, i.e. $SL(n, C)$, $SU(p, q)$, and $Sp(n, C)$, but was unable
to manage the group $G_2$ using his methods. For a converse and related results, the reader is referred to Appendix 1 of the paper [2] of the author and J. J. F. Fournier.

Here we shall give a new proof for the case where $G$ is $SL(2, R)$ and indicate how it differs from that of Kunze and Stein, and, at the end, discuss very briefly the general case. The general case will be proved in detail elsewhere.

**The Proof for $SL(2, R)$.** - Now we write $G$ for $SL(2, R)$; every element $g$ of this group is a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of real numbers of determinant one. The fractional linear transformation $r \mapsto g(r)$ is defined by the formula:

$$g(r) = \frac{ar + c}{br + d} \quad r \in R;$$

its modulus $dg(r)/dr$ is given by the formula

$$\frac{dg(r)}{dr} = |br + d|^{-2}.$$

The class-one principal series of $G$ is the set of representations $\pi_z$, where $z$ is a complex number, which act on the space of measurable functions on $R$ (strictly speaking, classes of measurable complex-valued functions defined almost everywhere) according to the rule

$$\pi_z(g) \xi(r) = |br + d|^{-1-z} \xi(g(r)) \quad r \in R.$$

It is easily checked that $\pi_z$ acts isometrically on $L^q(R)$ when

$$q \cdot \text{Re}(z) = (2 - q).$$

For the images under $\pi_z$ of two functions equal almost everywhere are equal almost everywhere, and

$$\int_R dr |\pi_z(g) \xi(r)|^q = \int_R dr |br + d|^{-2} |\xi(g(r))|^q$$

$$= \int_R dg(r) |\xi(g(r))|^q.$$