Type I Censoring and the Structural Approach

1. Introduction

In a recent paper [1], Whitney and Minder criticized some aspects of two papers [2], [3] by K.V. Bury and the author of this paper. We had applied the structural model to life testing situations involving the exponential and two parameter Weibull distributions. They noted, correctly, that the structural model does not apply with type I censoring. They agreed that it does apply for type II censoring [4], and that our results for this case, although not obtained directly by a structural argument, were correct.

Statistical inference from Type I censored data is generally difficult (because the number of observations is a random variable (r.v.)). The purpose of this paper is to show that although the structural model is not applicable, some of the concepts of the method can be used to develop helpful inferences from type I censored data that are not ordinarily available by other methods. Only the simplest case of Type I censoring will be considered: n items are put on test, the failure times are recorded, and the experiment is terminated after a fixed time T. The life times are assumed to be distributed according to \((1/\sigma_0)\hat{f}(x/\sigma_0)\), \(\sigma_0 > 0\).

Using structural methods, it will be shown how, from Type I censored data, tests of significance of hypothesized values of \(\sigma_0\) may be performed, confidence intervals for \(\sigma_0\) may be constructed, maximum likelihood estimates \(\hat{\sigma}_0\) for \(\sigma_0\) may be obtained, and a new random variable \(\bar{\gamma}\) defined, whose probability distribution, depending on \(\sigma_0\), some readers might wish to use to assess possible values of \(\sigma_0\) and possible values of the reliability function for a population of life times distributed as \((1/\sigma_0)\hat{f}(x/\sigma_0)\). In addition, "prediction" distributions will be developed for \(N\) future observations from such a population, for the largest and smallest of \(N\) future observations, and for the number of failures among a given number of items in a given time. Finally, results will be given for the special case of an exponential distribution, and a numerical example treated.

It is assumed that the reader is familiar with structural inference. For a recent elementary introduction, see Bernholtz and Tan [5].
2. The Model and Some Associated Geometry

The data consist of \( k \) ordered observations \( 0 < t_1^O < t_2^O < \ldots < t_k^O < T \). \( t^O = (t_1^O, \ldots, t_k^O) \) is related to \( \sigma^0 \) by the equation \( t^O = \sigma^0 e^O \). \( e^O = (e_1^O, \ldots, e_k^O) \) is a realized but unknown value of a r.v. \( E \), while \( k \) is a known value of a r.v. \( K \). The probability model for this experiment is the mixed joint probability density function/probability mass function

\[
f(e_1, \ldots, e_j; n, T, \sigma^O) = \frac{f(e_1) \cdots f(e_j)}{{j \choose n} \left( F(T/\sigma^0) \right)^j \left( 1 - F(T/\sigma^0) \right)^{n-j}}
\]

where \( 0 < e_1 < e_2 < \ldots < e_j < T/\sigma^0 \) and \( j = 0, 1, \ldots, n \). For the analysis of this paper it is enough to consider \( j = k \) only which results in a considerable simplification of notation without affecting the argument. Thus the discussion will deal with \( f(e, k; \sigma^O) = f(e_1, \ldots, e_k; k; n, T, \sigma^O) \) on the open set

\[
\mathcal{X} = (T/\sigma^0) = \{ x = (x_1, \ldots, x_k) : 0 < x_1 < \ldots < x_k < T/\sigma^0 \}
\]

Here

\[
F(T/\sigma^0) = \int_0^{T/\sigma^0} f(u) \, du
\]

and \( t = (t_1, \ldots, t_k) = \sigma^0 e = \sigma^0 (e_1, \ldots, e_k) \).

The sets \( \mathcal{X} \cap \{ ax : x \in \mathcal{X}, a > 0 \} \) constitute a partition of \( \mathcal{X} \). Let \( x \) be any point in \( \mathcal{X} \). Clearly \( y_k^{-1} y = r(y) = (r_1(y), \ldots, r_k(y)) \) is a fixed point for all \( y \) in \( \mathcal{X} \cap \{ ax : a > 0 \} \). Note that \( 0 < r_1(x) < r_2(x) < \ldots < r_k(x) = 1 \). Any point \( y \) in \( \mathcal{X} \cap \{ ax : a > 0 \} \) can be written as \( y = y_k r(x), \ 0 < y_k < T/\sigma^0 \).

Let \( \mathcal{A} = \{(r_1, \ldots, r_k) : 0 < r_1 < r_2 < \ldots < r_k = 1 \} \). \((0, T/\sigma^0) \times \mathcal{A} \) will denote the cartesian product of the real (open) interval \((0, T/\sigma^0)\) and the (open) set of points \( \mathcal{A} \).

3. Conditional and Marginal Densities

Let \( e \in \mathcal{X} \). The representation \( e = e_k r(e) \) defines a one-to-one mapping \( e \mapsto (e_k, r(e)) \) on \( \mathcal{X} \) onto \((0, T/\sigma^0) \times \mathcal{A} \) thus defining a new random variable \((E_k, R)\) with values \((e_k, r)\) in \((0, T/\sigma^0) \times \mathcal{A} \). Let \( f^*(e_k, r, k; \sigma^0) \) denote the mixed probability density/probability mass of \((e_k, r, k)\) with respect to an appropriately chosen measure \( M \) on \((0, T/\sigma^0) \times \mathcal{A} \). Suppose \( \Delta \) is an "elemental" set in \( R^+ \) (the set of positive real numbers) containing the point \( 1 \), and \( \rho \) is an elemental set in \( \mathcal{A} \) containing the point \( r \). Let \( e_k \Delta = \{ e_k b : b \in \Delta \} \) and \( \Delta \rho = \{ br' : b \in \Delta, r' \in \rho \} \). The probability of the event \((e_k \Delta) \times \rho \times \{ k \} \) in \((0, T/\sigma^0) \times \mathcal{A} \times \{ k \} \) equals the probability of the event \((e_k \Delta) \rho, k) \) in \( \mathcal{X} \times \{ k \} \) where \((e_k \Delta) \rho, k) \) in \( \mathcal{X} \times \{ k \} \). Hence,