Summary

It is well known that, for a gaussian process to be stationary, it is necessary and sufficient that the infinite order autocovariance matrix should be positive definite. This fact can be used to obtain the stationarity conditions, for a general autoregressive process; and, hence, the stationarity and invertibility conditions, for any mixed autoregressive moving average process. An interesting connection with a recently reported recursive approach is also noted.

(Autoregressive, Moving Average and Mixed Processes; Stationarity and Invertibility; Gaussian Process; Autocovariance Matrix; Positive Definiteness)

1. Introduction

Consider a gaussian white noise sequence \( \{ \xi_i \} \), whose elements are all identically, but independently distributed as \( N(0,1) \). A second sequence \( \{ z_i \} \) can be generated from \( \{ \xi_i \} \) according to

\[
\sum_{j=0}^{p} \phi_j z_{i-j} = \sum_{j=0}^{q} \theta_j \xi_{i-j}
\]

(1)

where \( \phi_0 = \theta_0 = 1 \) and \( \phi_p, \theta_q \neq 0 \). (1) is then termed a mixed autoregressive moving average process of order \( (p,q) \). Putting \( p = 0 \) or \( q = 0 \), we get 'pure' models called, respectively, a moving average process of order \( q \) or an autoregressive process of order \( p \).

The polynomials, associated with (1)

\[
\sum_{j=0}^{p} \phi_j \zeta^j \quad \text{and} \quad \sum_{j=0}^{q} \theta_j \zeta^j
\]

(2)

in the complex variable \( \zeta \) are termed, respectively, the autoregressive and moving average polynomials (of order \( p \) and \( q \)). We will interest ourselves in stationary and invertible processes, for which the polynomials (2) have all their zeros strictly outside the unit circle.
The autocovariance at lag $k$ of $\{Z_i\}$ is then defined by

$$\gamma_k = E[Z_i Z_{i-k}]$$

and the autocovariance matrix, of order $k$, is defined by

$$\tau_k = (\tau_{rs})$$

where

$$\tau_{rs} = \gamma_{|r-s|}. $$

For stationarity, it is necessary and sufficient that $\tau_\infty$ should be positive definite.

Finally, consider the polynomial

$$g(\zeta) = \sum_{j=0}^{n} g_j \zeta^j \quad (3)$$

of degree $n > 0$, in the complex variable $\zeta$, with real coefficients $g_0, g_1, \ldots, g_n$ subject to $g_0 \neq 0, g_n \neq 0$. Then Anderson [1975] has shown that necessary and sufficient conditions for the zeros of (3) to lie strictly outside the unit circle are that

$$|g_n| < |g_0| \quad (4)$$

and the reduced polynomial

$$g^*(\zeta) = \sum_{j=0}^{n-1} (g_0 g_j - g_n g_{n-j}) \zeta^j \quad (5)$$

has all its zeros strictly outside the unit circle.

2. Stationarity Conditions for Autoregressive Processes

Now $\tau_k$ is positive definite if and only if $\tau_k^{-1}$ is positive definite. But it is well known (for instance, see Anderson [1976]) that for a $p$th order autoregressive process $\tau_k^{-1}$ (see p.51) is a $k \times k$ centrosymmetric matrix. Then, for $k > 2p$, the determinant of $\tau_k^{-1}$ is easily reduced to the $p \times p$ centrosymmetric determinant, $\Delta_p$, with $(r,s)^{th}$ element = $\pi_{rs}$, where, for $1 \leq r, s \leq p$,

$$\pi_{rs} = \sum_{h=1}^{p} (\phi_{r-h} \phi_{s-h} - \phi_{p-r+h} \phi_{p-s+h}) \quad (6)$$

and, when $j < 0$ or $j > p$, $\phi_j = 0$. 50