ON THE RECURSIVE SEQUENCE $x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n}$

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Abstract. The boundedness, global attractivity, oscillatory and asymptotic periodicity of the positive solutions of the difference equation of the form

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n}, \quad n = 0, 1, \ldots$$

is investigated, where all the coefficients are nonnegative real numbers.

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1. Introduction

Recently there has been a great interest in investigating nonlinear difference equations, see, for example, [1-9] and the references therein. The following conjecture was formulated in [5]:

Conjecture 5.2.4 Every positive solution of the difference equation

$$x_{n+1} = \beta + \frac{x_{n-1}}{x_n}, \quad n = 1, 2, \ldots$$

is bounded if and only if $\beta \geq 1$. Furthermore, if $\beta = 1$, then every positive solution converges to a period two solution and, if $\beta > 1$, then every positive solution converges to the equilibrium $\beta + 1$.

A first discussion of the problem is given in [2]. Motivated by [2], in [3] the authors have considered the equation

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n}, \quad n = 0, 1, \ldots, \quad (1)$$
with positive initial conditions \( x_{-1} \) and \( x_0 \), under the assumptions \( \alpha \in [0, \infty) \) and \( p \in [1, \infty) \). They have investigated local stability, oscillation and boundedness character of the positive solutions of Eq.(1).

The following four results, which we incorporate into a theorem, were proved in [3].

**Theorem A.** Consider Eq.(1). Then the following statements are true:

1. The equilibrium point \( \bar{x} = \alpha + 1 \) of Eq.(1) is locally asymptotically stable if \( \alpha > 2p - 1 \) and unstable if \( \alpha \in [0, 2p - 1) \).
2. Let \( (x_n) \) be a positive solution of Eq.(1) which consists of a single semi-cycle. Then \( (x_n) \) converges monotonically to \( \bar{x} \).
3. Let \( (x_n) \) be a positive nonequilibrium solution of Eq.(1) which consists of at least two semi-cycles. Then \( (x_n) \) is oscillatory. Moreover, every semi-cycle has length one except possibly the first semi-cycle.
4. Let \( \alpha \in [0, 1) \) and let \( (x_n) \) be a solution of Eq.(1) such that \( 0 < x_{-1} \leq 1 \) and \( x_0 \geq 1/(1 - \alpha)^{1/p} \). Then \( \lim_{n \to \infty} x_{2n} = \infty \) and \( \lim_{n \to \infty} x_{2n+1} = \alpha \).

The first result is a simple consequence of the linearized stability theorem [4]. The second one is also simple and it can be generalized as follows:

**Theorem 1.** Consider the difference equation

\[
x_{n+1} = \alpha + f \left( \frac{x_n - k}{x_n} \right), \quad n = 0, 1, ..., \quad (2)
\]

with positive initial conditions \( x_{-k}, ..., x_0 \), where \( \alpha \in [0, \infty) \), \( k \in \mathbb{N} \) and the function \( f \) is nonnegative, strictly increasing on the interval \([0, \infty)\) and \( f(1) = 1 \). Let \( (x_n) \) be a nonoscillatory solution of Eq.(2). Then \( (x_n) \) converges to \( \bar{x} = \alpha + 1 \).

**Corollary 1.** Consider Eq.(2) with \( k = 1 \), positive initial conditions \( x_{-1} \) and \( x_0 \), where \( \alpha \in [0, \infty) \) and the function \( f \) is nonnegative, strictly increasing on the interval \([0, \infty)\) and \( f(1) = 1 \). Let \( (x_n) \) be a solution of the equation which consists of a single semi-cycle. Then \( (x_n) \) converges monotonically to \( \bar{x} = \alpha + 1 \).

It was not shown in [3] the existence of solutions of Eq.(1) which consist of a single semi-cycle. So we leave to the reader who are interested in this area the following open problem.

**Open problem 1.** Investigate whether or not there is a nonequilibrium solution of Eq.(1) which consists of a single semi-cycle.

The third result in Theorem A is a special case of Theorem 3.2 in [4]. The proof of the last result in Theorem A follows the lines of the proof of Theorem 3.1 in [2]. This caused that they repeated the same mistake which