On Hecke Eigenforms of Degree $n$

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1 Introduction and statement of result

Let $S_k(\Gamma_n)$ be the complex vector space of Siegel cusp forms of weight $k \in \mathbb{Z}$ with respect to the Siegel modular group $\Gamma_n := \text{Sp}(n; \mathbb{Z})$ ($n \in \mathbb{N}$). For $F \in S_k(\Gamma_n)$ we denote by $a(T)$ ($T$ a half-integral, positive definite $n$-rowed matrix) the Fourier coefficients of $F$. If $F \in S_k(\Gamma_n)$ is a Hecke eigenform, we let

$$Z_F(s) := \prod_p Z_{F,p}(p^{-s})^{-1} \quad (\text{Re } s > 0)$$

be the spinor zeta function of $F$, where

$$Z_{F,p}(Y) = (1 - \alpha_{p,0} Y) \prod_{1 \leq i_1 < \cdots < i_j \leq n} \left(1 - \alpha_{p,0} \alpha_{p,i_1} \cdots \alpha_{p,i_j} Y \right)$$

is the local spinor polynomial and $\alpha_{p,0}, \ldots, \alpha_{p,n}$ are the Satake $p$-parameters of $F$. It is conjectured that $Z_F(s)$ (completed with $\Gamma$-factors) can be meromorphically continued to the whole complex plane and satisfies a functional equation with respect to $s \mapsto nk - \frac{n(n+1)}{2} + 1 - s$. This is only known for $n \leq 2$ (cf. [1]).

The purpose of this paper is to show the following

**Theorem.** Let $n \geq 3$ and $F, G \in S_k(\Gamma_n)$ be two Hecke eigenforms with Fourier coefficients $a(T)$ and $b(T)$, respectively. Suppose that $Z_F(s)$ and $Z_G(s)$ have a meromorphic continuation to $\mathbb{C}$ and satisfy the conjectured functional equation. Suppose that $a(mT) = b(mT)$ for every primitive matrix $T$ and every $m \in \mathbb{N}$ with $v_p(m) \leq 2^n - 2$ for every prime $p$, where $v_p(m)$ denotes the usual $p$-adic exponent of $m$. Then $F = G$.

The above theorem is a generalization of an analogous result in the case $n = 2$ (cf. [2]). Note that in this case the exponent of $m$ can be improved to be one. Also, as mentioned above, no additional information on the spinor zeta functions of $F$ and $G$ is needed.

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Remark. H. KATSURADA kindly informed the authors that in his paper "On the coincidence of Hecke eigenforms" (appeared in this volume, pp. 77–83) he has proved the same result as given in [2] for $n = 2$ for arbitrary $n \geq 2$. No unproved hypothesis for spinor zeta functions is needed there.

2 Proof of theorem

The proof essentially follows the same pattern as the one given in [2]. In addition, however, we have to make use of relations given by ŽARKOVSKAJA between eigenvalues and Fourier coefficients for arbitrary $n$ ([9]). We also make use of (part of) the converse theorem for arbitrary $n$ due to WEISSAUER (cf. [7], [8]).

Lemma 1. Let $F \in S_k(\Gamma_n) \setminus \{0\}$ with Fourier coefficients $a(T)$. Then there exists a Groessen character $\varphi$ on $\text{SL}(n; \mathbb{R})$ (in the sense of Maaß) such that the twisted Maaß–Koecher series

$$D_{F,\varphi}(s) := \sum_{T > 0/\sim} \frac{a(T)\varphi((\det T)^{-1/n} T)}{\epsilon(T) \det(T)^s} \quad (\text{Re } s \gg 0)$$

does not vanish identically (the sum extends over a complete system of representatives of $\text{SL}(n; \mathbb{Z})$-classes of half-integral, positive definite matrices $T$ and $\epsilon(T) := \sharp\{U \in \text{SL}(n; \mathbb{Z}) : U^t TU = T\}$).

Proof. [7], cf. also [8]. \qed

Lemma 2. Let $F, G \in S_k(\Gamma_n)$ be two Hecke eigenforms with Fourier coefficients $a(T)$ resp. $b(T)$. Let $N$ be a fixed $n$-rowed half-integral, positive definite matrix and suppose that $a(mN) = b(mN)$ for every $m \in \mathbb{N}$ with $v_p(m) \leq 2^n - 2$ for every prime $p$. Then

$$Z_F^{-1}(s) \sum_{m=1}^{\infty} \frac{a(mN)}{m^s} = Z_G^{-1}(s) \sum_{m=1}^{\infty} \frac{b(mN)}{m^s} \quad (\text{Re } s \gg 0).$$

Proof. Let

$$Q(X_0, X_1, \ldots, X_n, Y) = (1 - X_0 Y) \prod_{1 \leq i_1 < \cdots < i_j \leq n} (1 - X_0 X_{i_1} \cdots X_{i_j} Y)$$

be the local spinor polynomial, viewed as an element of $\mathbb{C}[X_0^\pm, \ldots, X_n^\pm]^{W_n}[Y]$ where $W_n$ is the Weyl group. For a fixed prime $p$ the local Hecke-Algebra $L_p^n$ may be identified with $\mathbb{C}[X_0^\pm, \ldots, X_n^\pm]^{W_n}$ via the following isomorphism (cf. [4], p. 258, note the different normalization), given on the generators of $L_p^n$ ([4], p. 250)