Sixteen-dimensional Locally Compact Translation Planes Admitting $SU_4 \mathbb{C} \cdot SU_2 \mathbb{C}$ or $SU_4 \mathbb{C} \cdot SL_2 \mathbb{R}$ as a Group of Collineations

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1 Introduction

In this paper, all 16-dimensional locally compact translation planes admitting the almost direct product $SU_4 \mathbb{C} \cdot SU_2 \mathbb{C}$ or $SU_4 \mathbb{C} \cdot SL_2 \mathbb{R}$ as a group $\Delta$ of collineations will be determined explicitly. The classical octonion plane $\mathcal{P}_2 \mathbb{O}$ belongs to each of these two families of planes, but there is a multitude of other planes, see the Classification Theorem 4.4.

The full collineation group $G$ of such a plane is a Lie group whose dimension will be shown to be 35 if the plane is not isomorphic to $\mathcal{P}_2 \mathbb{O}$; in fact, the connected component $G^1$ of $G$ is a semidirect product of the 16-dimensional translation group, the 1-dimensional group of real homotheties and the 18-dimensional group $\Delta$.

Except for the octonion plane, the group $G^1$ has precisely one fixed line (the translation axis) and no fixed point. In [9], it will be shown that there are just three families of 16-dimensional locally compact translation planes in which $G$ has this property and satisfies $\dim G \geq 35$, namely the two families of planes considered here and a subfamily of the $SL_2 \mathbb{H}$-planes determined by H. Löwe [12]; the latter planes satisfy $\dim G = 35$, as well ($\mathcal{P}_2 \mathbb{O}$ excepted). In fact, even within the larger class of all 16-dimensional compact projective planes in general, the planes described here and the mentioned $SL_2 \mathbb{H}$-planes are the only planes whose automorphism group has a closed subgroup of dimension at least 35 fixing no point and precisely one line, since by a recent result of H. Salzmann [15] every such plane is a translation plane.

The plan of the paper is as follows: In Section 2, we characterize the subgroups of $SU_4 \mathbb{C}$ isomorphic to $U_2 \mathbb{H}$, which will be important later as stabilizers of lines. In Section 3, we determine how a group locally isomorphic to $SU_4 \mathbb{C} \times SU_2 \mathbb{C}$ or $SU_4 \mathbb{C} \times SL_2 \mathbb{R}$ can act as a group of collineations of a 16-dimensional locally compact translation plane, and we study this action. In Section 4, all these planes are explicitly constructed. In Section 5, we determine all collineations and possible isomorphisms of these planes.

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2 Large subgroups of SU\textsubscript{4}\mathbb{C}

2.1 Quaternion coordinates. A subgroup of SU\textsubscript{4}\mathbb{C} isomorphic to U\textsubscript{2}\mathbb{H} can be seen easily if one identifies \mathbb{C}\textsuperscript{2} with the quaternion field \mathbb{H} in the following way: Let 1, i, j, k be the usual basis of \mathbb{H} as an \mathbb{R}-algebra, and consider \mathbb{C} as the subfield \mathbb{R} + \mathbb{R}i of \mathbb{H}. Then \mathbb{H} is a right \mathbb{C}-vector space of dimension 2 with basis 1, j, and as such we can identify \mathbb{C}\textsuperscript{2} with \mathbb{H} via the map
\[
\mathbb{C}^2 \rightarrow \mathbb{H} : \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \mapsto c_1 + j c_2.
\]
There results an identification of \mathbb{C}^4 with \mathbb{H}^2,
\[
\mathbb{C}^4 \rightarrow \mathbb{H}^2 : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + j x_2 \\ x_3 + j x_4 \end{pmatrix} = x \in \mathbb{H}^2,
\]
under which the canonical basis vectors \textbf{e}_1, \textbf{e}_2, \textbf{e}_3, \textbf{e}_4 of \mathbb{C}^4 correspond to the following vectors of \mathbb{H}^2:
\[
\textbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \textbf{e}_2 = \begin{pmatrix} j \\ 0 \end{pmatrix} = \textbf{e}_1 j, \quad \textbf{e}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \textbf{e}_4 = \begin{pmatrix} 0 \\ j \end{pmatrix} = \textbf{e}_3 j.
\]
For the vector \( x \in \mathbb{H}^2 \) in (1), we form the scalar multiple \( xj \) in the right \mathbb{H}-vector space \mathbb{H}^2 and calculate the coordinates of \( xj \) in \mathbb{C}^4 according to the identification (1). Using the fact that \( cj = j \bar{c} \) for \( c \in \mathbb{C} \), we find that
\[
xj = \begin{pmatrix} -x_2 + j x_1 \\ -x_4 + j x_3 \end{pmatrix} \cong \begin{pmatrix} \bar{x}_2 \\ \bar{x}_1 \\ \bar{x}_4 \\ \bar{x}_3 \end{pmatrix}.
\]
In particular, \( x \) and \( xj \), when considered as elements of \( \mathbb{C}^4 \), are orthogonal with respect to the standard hermitian form.

Via the identification (1), the unitary quaternion group of the right \mathbb{H}-vector space \mathbb{H}^2 can be described as a subgroup of SU\textsubscript{4}\mathbb{C} by
\[
U_2\mathbb{H} = SU_4\mathbb{C} \cap GL_2\mathbb{H}.
\]

2.2 Lemma Let \( H \) be a proper closed connected subgroup of SU\textsubscript{4}\mathbb{C} of dimension at least 8. Then \( H \) is conjugate to one of the following subgroups:

(i) the subgroup \( U_2\mathbb{H} \) introduced above
(ii) the stabilizer of a non-zero vector in \( \mathbb{C}^4 \); these stabilizers are mutually conjugate and isomorphic to SU\textsubscript{3}\mathbb{C}
(iii) the stabilizer of a 3-dimensional subspace of \( \mathbb{C}^4 \); these stabilizers are mutually conjugate and isomorphic to U\textsubscript{3}\mathbb{C}.

Proof. 1) First assume that \( H \) contains a central torus \( T \). An infinitesimal generator \( A \in \mathbb{C}^{4\times 4} \) of a one-parameter subgroup of \( T \) is a skew hermitian matrix of trace 0 (those are the elements of the Lie algebra of SU\textsubscript{4}\mathbb{C}). In particular, \( \mathbb{C}^4 \) has an orthonormal basis of eigenvectors of \( A \) belonging to at least two different eigenvalues,