Generalizations of Girstmair's Formulas

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Abstract. In 1994 and 1995 Girstmair gave (relative) class number formulas for the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$, $p$ an odd prime with $p \equiv 3 \pmod{4}$ and $p \geq 7$, using the coefficients of the digit expression of $1/p$ and $z/p$, respectively, where $z$ is an integer with $1 \leq z \leq p - 1$. We extend the formulas to an imaginary abelian number field.

1 Introduction

In his papers [3] and [4] Girstmair gave the following formulas (2) and (4) for the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$, $p$ an odd prime with $p \equiv 3 \pmod{4}$ and $p \geq 7$. To describe the formulas he used the coefficients of the digit expression of $1/p$ and that of $z/p$, respectively, where $z$ is an integer with $1 \leq z \leq p - 1$:

Theorem 1. (Girstmair [3]) Let $p \geq 7$ be an odd prime with $p \equiv 3 \pmod{4}$ and $g \geq 2$ be a primitive root modulo $p$. Let

$$\frac{1}{p} = \sum_{k=1}^{\infty} \frac{x_k}{g^k}, \quad x_k \in \{0, 1, \ldots, g-1\}. \tag{1}$$

Then we have

$$(g + 1)h_K^* = (x_2 + x_4 + \cdots + x_{p-1}) - (x_1 + x_3 + \cdots + x_{p-2}), \tag{2}$$

where $h_K^*$ is the relative class number of $K = \mathbb{Q}(\sqrt{-p})$.

Theorem 2. (Girstmair [4]) Let $p$, $K$ and $h_K^*$ be as in Theorem 1. Take integers $b$ with $b \geq 2$ and $z$ with $1 \leq z \leq p - 1$. Let

$$\frac{z}{p} = \sum_{k=1}^{\infty} \frac{c_k}{b^k}, \quad c_k \in \{0, 1, \ldots, b-1\}. \tag{3}$$

Suppose that the sequence $\{c_k\}$ has period $(p - 1)/2$. Then we have

$$c_1 + c_2 + \cdots + c_{(p-1)/2} = \frac{z(b-1)(p-1)}{4} - \left(\frac{z}{p}\right) \frac{b-1}{2} h_K^*, \tag{4}$$

where $\left(\frac{z}{p}\right)$ is the Legendre symbol.
As will be shown, the formulas (2) and (4) are obtained by taking the parameter \( b \) as \( b = g \) and \( b = g^2 \pmod{p} \) in the formula (23), respectively, while DIRICHLET's class number formula for the imaginary quadratic field \( \mathbb{Q}(-p) \) with odd prime conductor \( p \) and CARLITZ and OLSON's formula for the \( p \)-th cyclotomic field in [2] as \( b = p + 1 \), and HAZAMA's formula for the \( p \)-th cyclotomic field in [7] as \( b = 2 \) (See [8], [9] and [10]).

In this paper we generalize the formulas (2) and (4) to an imaginary abelian number field. To compare our generalized formulas with the GIRSTMAIR's ones, we give some formulas for imaginary abelian number fields with prime-power conductor in Sections 2 and 3. In Section 4, as examples, we give some formulas for composite cyclotomic fields. In Section 5 we give our generalized formulas, which generalize all the formulas in Sections 1 to 4. In Section 6 we prove the generalized formulas.

Our results would be an answer to the question of GIRSTMAIR [5]. Some of the formulas in Sections 2 and 3 have already introduced in [9].

Throughout this paper we denote by \( h_K^* \) the relative class number of an imaginary abelian number field \( K \) and by \( w_K \) the number of roots of unity in \( K \).

2 Formulas for imaginary abelian number fields with odd prime-power conductor

First we generalize the formula (2): Let \( p^a \) be a power of an odd prime \( p \) and \( g \geq 2 \) a primitive root modulo \( p^a \). Let

\[
\frac{1}{p^a} = \sum_{k=1}^{\infty} \frac{x_k}{g^{k}}, \quad x_k = x(k) \in \{0, 1, \ldots, g-1\},
\]

which we call the digit expression of \( 1/p^a \) with respect to the basis \( g \). Then we have

\[
x(k) = \left\lfloor \frac{g R_{p^a}(g^{k-1})}{p^a} \right\rfloor
\]

for \( k = 1, 2, \ldots \), where \( R_{p^a}(x) \) is the least positive residue of an integer \( x \) modulo \( p^a \) (See [4, Satz 2]). Since \( x(k + \varphi(p^a)) = x(k) \) for any \( k \geq 1 \), where \( \varphi \) is the Euler's totient function, we extend the definition of \( x(k) \) by

\[
x(k + \varphi(p^a)) = x(k)
\]

for every integer \( k \).

As a generalization of Theorem 1 we have

**Theorem 3.** Let \( K \) be an imaginary abelian number field of degree \( 2n = [K : \mathbb{Q}] \) with odd prime-power conductor \( p^a \), \( p \) an odd prime. Let the sequence \( \{x(k)\} \) be