Nonuniform Lowness and Strong Nonuniform Lowness

Li Hongzhou (李宏宙) and Li Guanying (李冠英)

Department of Computer Science, South China Normal University, Guangzhou 510631

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Abstract

The concepts of the nonuniform and strong nonuniform lownesss are introduced. Those notions provide a uniform framework to study connections between the polynomial-time hierarchy and sparse sets.

Keywords: Nonuniform and strong nonuniform lowness, polynomial-time hierarchy, sparse sets, relativizations.

1 Introduction

A central problem in computational complexity theory is whether or not the polynomial time hierarchy collapses. There have been various approaches to this problem. Using sparse set to study this problem has been very active.

People have attempted to reduce uniform complexity to nonuniform complexity. Mahaney[8] showed that the existence of a sparse $\leq^p_m$-complete set for $NP$ implies $P = NP$. Karp and Lipton[4] showed that sparse set cannot be $\leq^p_T$-complete for $NP$ unless the polynomial-time hierarchy collapses $\Sigma^p_2$. Yap[13] showed that if the nonuniform polynomial-time hierarchy collapses at level $n > 0$, i.e., $\Sigma^p_{n+1} \subseteq \Sigma^p_n / poly$, then the polynomial-time hierarchy collapses at level $n + 2$, i.e., $\Sigma^p_{n+3} = \Sigma^p_{n+2}$. This strengthens a generalization of the result of Karp and Lipton. Recently, Ogihara and Watanabe[9] improved Mahaney’s result by showing that sparse set cannot be $\leq^p_{MT}$-hard for $NP$ unless $P = NP$.

On the other hand, people have studied the problem by considering relativization with respect to sparse oracles. Balcazar et al.[1] showed that the polynomial-time hierarchy collapses if and only if for every sparse set $S$ the polynomial-time hierarchy relative to $S$ collapses (see also [7]) if and only if there exists a sparse set $S$ such that the polynomial-time hierarchy relative to $S$ collapses. Furthermore, Li[8] obtained the similar results by considering the oracle set $A \in PH/poly$.

In [1,7], Balcazar et al. have indicated that their results are of a different type from those of Karp and Lipton[4] and Yap[13].

In this paper, we look at the above results under a new point of view. The concepts of the nonuniform and strong nonuniform lowness are introduced. Those
notions provide a uniform framework to study connections between the polynomial
time hierarchy and sparse sets. More precisely:

In Section 2, we define our notions. Let $SL(B) = \{A \mid NP(A) \subseteq NP(B) / \text{poly}\}$
and $STL(B) = \{A \mid A \in (NP(B) \cap Co - NP(B)) / \text{poly}\}$. A set $A$ is nonuniform
low relative to $B$ if $A \in SL(B)$ and $A$ is strong nonuniform low relative to $B$ if $A \in
STL(B)$. We show that $SL(B) = NP(B) / \text{poly} \cap Co - NP(B) / \text{poly}$. For each $n > 0$,
define $SL(n,n) = \{A \mid \sum_n^P(A) \subseteq \sum_n^P / \text{poly}\}$ and $STL(n,n) = \{A \mid \sum_{n-1}^P(A) \subseteq
(\sum_n^P \cap \prod_n^P) / \text{poly}\}$. A set $A$ is nonuniform low if $A \in SL(n,n)$ and $A$ is strong
nonuniform low if $A \in STL(n,n)$.

In Section 3, our main results are proved. First, we show (Theorem 1)
if $A$ is nonuniform low relative to $B$, then $\sum_2^P(A) \subseteq \sum_2^P(A \oplus SAT_B)$
by applying the quantifier-coding methods introduced by Ko and Schöning[5] with new flavor.
From this result, the well-known result of Yap[13] and the main results of Balcázar
et al.[1] follows immediately, thus their results can be unified and simplified under
the nonuniform lowness. To our surprise, the proof of our result does not depend on
"The Notion of the Self-Reducibility". Secondly, we show (Theorem 2) that if $A$ is
self-reducible and $A \in STL(B)$, then $\sum_2^P(A) \subseteq \sum_2^P(B)$. From this result, it follows
that for every $n > 0$ if $\sum_n^P \subseteq (\sum_n^P \cap \prod_n^P) / \text{poly}$, then the polynomial-time hierarchy
collapses at level $n + 1$, i.e., $\sum_{n+2}^P = \sum_{n+1}^P$. This improves the result of Yap[13] and
strengthens a generalization of the result of Karp and Lipton[4].

2 Preliminaries

2.1 Notation

It is assumed that the reader is familiar with the basic concepts and results in
[3,12].

All our sets will be languages over the fixed alphabet $\Sigma = \{0,1\}$. For a set $A$ and
$n \geq 0$, define $A_{=n} = \{x \in A \mid |x| = n\}$, $A_{\leq n} = \{x \in A \mid |x| \leq n\}$, where $|x|$ is the
length of $x$, and $\overline{A} = \sum^* - A$. $|S|$ denotes the cardinality of the set $S$. The join
of two sets $A$ and $B$ is $A \oplus B = \{0x \mid x \in A\} \cup \{1x \mid x \in B\}$. For a class $D$ let
$Co - D$ be the class of complements of sets in $D$. Let $(\cdot,\cdot)$ be a pairing function.
This function and its inverse are computable in polynomial time. For all $k > 2$ and
for all $y_1,y_2,\ldots,y_k$ let $(y_1, (y_2,\ldots,y_k))$ denote the string $(y_1, y_2,\ldots,y_k)$. A set $S$ is
sparse iff for some polynomial $p$ and each $n \geq 0$, $|S_{\leq n}| \leq p(n)$.

Our model of computation is the standard oracle Turing machine. Let $L(M)$
denote the set accepted by Turing machine, and $L(M,A)$ the set accepted by $M$
when using oracle set $A$. The class $P$ and $NP$ have their standard definition. Their
$A$-relativized versions are denoted $P(A)$ and $NP(A)$ respectively. The classes of the
(relativized) polynomial-time hierarchy $\sum_n^P(A)$ and $\prod_n^P(A)$ are defined inductively
as $\sum_0^P(A) = \prod_0^P(A) = P(A)$, $\sum_{n+1}^P(A) = NP(\sum_n^P(A))$, $\prod_{n+1}^P(A) = Co - \sum_{n+1}^P(A)$
for $n \geq 0$. If $A$ is the empty set, we simply write $\sum_n^P$ or $\prod_n^P$, respectively. The
following facts about the polynomial-time hierarchy will be used several times : if