A New Representation and Algorithm for Constructing Convex Hulls in Higher Dimensional Spaces

Lü Wei and Liang Youdong

CAD/CAM Research Center, Zhejiang University, Hanzhou 310027

Received January 17, 1990.

Abstract

This paper presents a new and simple scheme to describe the convex hull in \( \mathbb{R}^d \), which only uses three kinds of the faces of the convex hull, i.e., the \( d-1 \)-faces, \( d-2 \)-faces and 0-faces. Thus, we develop an efficient new algorithm for constructing the convex hull of a finite set of points incrementally. This algorithm employs much less storage and time than that of the previously-existing approaches. The analysis of the running time as well as the storage for the new algorithm is also theoretically made. The algorithm is optimal in the worst case for even \( d \).

1. Introduction

Computing the convex hull of a finite set of points is well studied in computational geometry, especially for two- and three-dimensional spaces \cite{1,2,3}. This is due to its wide applications in, for example, pattern recognition and image processing. Besides, there are many geometric problems that may be transformed into the convex hull problems \cite{4,5,6}.

In the higher dimensional cases, however, only two algorithms for constructing convex hulls are extensively recognized, i.e., the gift-wrapping (GW) method which proceeds from a facet to an adjacent facet in the guise in which one wraps a sheet around a plane-bounded object, and the beneath-beyond (BB) method which is based on an existing convex hull to construct a new one by adding a new point at a time. Since it has the on-line property and the performance comparable to that of the gift-wrapping one, the beneath-beyond method is very attractive to the users. Therefore, our interest here is to investigate the beneath-beyond technique.

While studying the data structure in the BB algorithm, we can see that the rub to design an efficient incremental algorithm of convex hulls depends mainly on the descriptions of the convex hull polytopes themselves. This is because the number of facets of a convex hull polytope is, at worst, exponential in the number of vertices, as possess serious difficulties in the representation of \( d \)-polytopes for large value of \( d \).

In the BB technique, a convex hull in \( \mathbb{R}^d \) is described as \((d+1)\)-lists which collect all the \( j \)-faces \((j = -1, \ldots, d-1)\) of the convex hull. Moreover, each record corresponding to a face contains a large number of different kind of pointers. It is quite easily seen that such a data structure is too complicated to use in practice \cite{1,2}. However, no simpler representation scheme for the convex hulls in higher dimensional spaces is carried out until now. Obviously, a simple description may reduce not only the space complexity but also the time complexity for searching and sorting in the implementations of the algorithm.

The aim of this paper is to present a new and simple scheme to describe the convex hull in \( \mathbb{R}^d \) so as to develop an improved BB algorithm for constructing the convex hull of a finite set of points.

2. Preliminaries

To serve as a foundation of the following discussions, some facts on the convex hull

---

This work is supported by the National Natural Science Foundation of China.
polytopes are recalled here.

Usually, this paper continues to use the notations and terminology introduced in [2] unless they are redefined. Let \( S_n = \{ P_i, \ i = 1, \ldots, n \} \) be a set of arbitrarily given \( n \) points in \( \mathbb{R}^d \), and denote the convex hull of the point set \( S_n \) by \( \text{ch}(S_n) \). Without loss of generality, we assume in the following sections that \( d \geq 2 \) and \( \dim(\text{aff}(S_n)) = d \), and denote the boundary of the convex hull \( \text{ch}(S_n) \) by \( \partial(\text{ch}(S_n)) \).

In this paper, we use “vertex”, “edge” and “facet” as the synonyms for 0-face, \( d-2 \)-face and \( d-1 \)-face of the convex hulls in \( \mathbb{R}^d \), respectively. It is worthwhile to note that the term “edge” used here is very different from that in [1, 2] where “edge” represents 1-face and the other terms were used for \( d-1 \)-face. In addition, we assume that the convex hull studied in this paper are the simplified \( d \)-polytopes [2]. Then, let us denote the sets of the facets and edges by \( F(S_n) \) and \( E(S_n) \), respectively, of the convex hull \( \text{ch}(S_n) \). Thus, each facet \( f \) in \( F(S_n) \) is a \( d-1 \)-simplex and each edge \( e \) in \( E(S_n) \) is a \( d-2 \)-simplex.

Besides, for an arbitrary set of \( J = \{ \delta_i \leq \mathbb{R}^d, \ i = 1, \ldots, j \} \), we denote \( \| J \| = \sum_{i=1}^{j} \delta_i \). From [1, 2], it follows that

**Lemma 1.** (i) \( \| F(S_n) \| = \partial(\text{ch}(S_n)) \).
(ii) for each edge \( e \) in \( E(S_n) \), there exist only two facets \( f_1 \) and \( f_2 \) in \( F(S_n) \) such that
\[
 f_1 \cap f_2 = e
\]

From the above lemma, it is known that \( \partial(\text{ch}(S_n)) \) is completely covered by the facets in \( F(S_n) \), and moreover, if \( P \) is in \( \partial(\text{ch}(S_n)) \) and not in \( \| E(S_n) \| \), then there is a unique facet \( f \) in \( F(S_n) \) such that \( P \) lies on \( f \). Therefore, it can be seen that the problem to construct the convex hull \( \text{ch}(S_n) \) is actually to compute and represent \( F(S_n) \) effectively.

For each facet \( f \) in \( F(S_n) \), denote the supporting hyperplane of \( f \) relative to \( \text{ch}(S_n) \) by \( H_f = \text{aff}(f) \), and from now on, suppose that \( \text{ch}(S_n) \) lies entirely in the close halfspace \( \{ P \in \mathbb{R}^d : H_f(P) \geq 0 \} \).

Now, from Lemma 1, we could deduce the classifications of the facets \( F(S_n) \) and the edges \( E(S_n) \) of \( \text{ch}(S_n) \) which depends on their relative position to a new point \( Q \). These classifications are defined as follows,
\[
 F^+(S_n, Q) = \{ f \in F(S_n) : H_f(Q) \geq 0 \};
\]
\[
 F^-(S_n, Q) = \{ f \in F(S_n) : H_f(Q) < 0 \};
\]
\[
 E^0(S_n, Q) = \{ e \in E(S_n) : \text{there exist } f_1 \text{ in } F^+(S_n, Q) \text{ and } f_2 \text{ in } F^-(S_n, Q) \text{ such that } f_1 \cap f_2 = e \}.
\]

When we refer to the classifications given in [2], \( F^+(S_n) \) is indeed the set of all yellow and blue facets, \( F^-(S_n) \) the set of all red facets, and \( E^0(S_n, Q) \) the set of all the \( d-2 \) faces with color orange or purple. Apparently, \( E^0(S_n, Q) \) is not an empty set if and only if the new point \( Q \) lies outside \( \text{ch}(S_n) \).

Now, for \( Q \) outside \( \text{ch}(S_n) \), we can construct a new set of \( d-1 \) simplices,
\[
 F^0(S_n, Q) = \{ f = (e, Q) : e \in E^0(S_n, Q) \},
\]
where \( f = (e, Q) \) denotes a \( d-1 \)-simplex whose vertices are the point \( Q \) and that of \( e \).

From the point of view of geometry [2], the set \( F^0(S_n, Q) \) can be interpreted intuitively as the set of facets of the supporting "cone" of \( \text{ch}(S_n) \) from \( Q \).

It is worthy of noting that since they are made only for the \( d-1 \)-faces and \( d-2 \)-faces, the above classifications are much simpler than that shown in [1, 2] where each kind of faces of the convex hull has to be classified. In the next two sections, we'll see that the above effective classifications enable us to achieve a simple description of the convex hulls in more than two dimensional spaces.

### 3. A Theorem on the Geometric Structure of the Convex Polytopes

Because we are interested in the on-line algorithms, the problem to be discussed now is to