A Type of Triangular Ball Surface and Its Properties

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Abstract

A new type of bivariate generalized Ball basis function on a triangle is presented for free-form surface design. Some properties of the basis function are given, then degree elevation, recursive evaluation and some other properties of the generalized Ball surfaces are also derived. It is shown that the proposed recursive evaluation algorithm is more efficient than those of the old surfaces.

Keywords: Triangular Ball surfaces, degree elevation, recursive evaluation.

1 Introduction

In the CONSURF system [1-3] developed by A. A. Ball, the basis functions for cubic polynomials were defined by

\[ B_0(u), B_1(u), B_2(u), B_3(u) = [(1 - u)^2, 2u(1 - u)^2, 2u^2(1 - u), u^2] \]

and hence cubic Ball curves with control points \( P_i \ (i = 0, 1, 2, 3) \) can be represented as

\[ P(u) = \sum_{i=0}^{3} B_i^3(u) P_i, \ 0 \leq u \leq 1. \]

Two types of higher degree generalized Ball basis functions and corresponding curves have been derived by Wang [4] and Said [5] independently. In Wang's paper, generalized Ball basis functions of arbitrary degree were defined as follows:

- if \( n \) is an even number

\[ B_i^n(u) = \begin{cases} (1 - u)^2 w^i & \text{for } 0 \leq i \leq \frac{n}{2} - 1 \\ w^2 & \text{for } i = \frac{n}{2} \\ u^2 w^{n-i} & \text{for } \frac{n}{2} + 1 \leq i \leq n \end{cases} \]

- if \( n \) is an odd number

\[ B_i^n(u) = \begin{cases} (1 - u)^2 w^i & \text{for } 0 \leq i \leq \frac{n-1}{2} \\ (1 - u) w^\frac{n+1}{2} & \text{for } i = \frac{n-1}{2} \\ w w^\frac{n+1}{2} & \text{for } i = \frac{n+1}{2} \\ u^2 w^{n-i} & \text{for } \frac{n+1}{2} \leq i \leq n \end{cases} \]

where \( w = 2u(1-u), \ 0 \leq u \leq 1. \) A parametric curve of degree \( n \), called a Wang-Ball curve, was represented as \( B_n(u) = \sum_{i=0}^{n} V_i B_i^n(u), \ 0 \leq u \leq 1. \) In Said's paper, the cubic Ball

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basis function (1) was extended to the polynomials of arbitrary odd degree and defined as

\[ A_n^i(u) = \begin{cases} 
\binom{n-1}{i} u^i (1 - u)^{\frac{n+1}{2}} & \text{for } 0 \leq i \leq \frac{n-1}{2} \\
\binom{n-1}{n-i} u^{n-i} (1 - u)^{\frac{n+1}{2}} & \text{for } \frac{n+1}{2} \leq i \leq n
\end{cases} \]

Hence, a Said-Ball curve of degree \( n \) (\( n \) is an odd number) can be represented by \( A_n(u) = \sum_{i=0}^n V_i A_n^i(u), \) \( 0 \leq u \leq 1. \) Note that, the curve \( A(u) \) reduces to a curve of degree \( n - 1 \) \((n - 1 \) is an even number) if \( P_{n-1} = P_{n+1}. \) So, Hu et al. suggested that the Said-Ball basis functions of arbitrary even degree should be defined additionally by

\[ A_n^i(u) = \begin{cases} 
\binom{n}{i} u^i (1 - u)^{\frac{n}{2}+1} & \text{for } 0 \leq i \leq \frac{n}{2} - 1 \\
\binom{n}{n-i} u^{n-i} (1 - u)^{\frac{n}{2}} & \text{for } i = \frac{n}{2} \\
\binom{n}{n-i} u^{n-i+1} (1 - u)^{n-i} & \text{for } \frac{n}{2} + 1 \leq i \leq n
\end{cases} \]

The properties of those two types of generalized Ball curves have been extensively investigated by Goodman and Said and Hu et al. Furthermore, a type of bivariate Ball basis function on a triangle has been proposed by Goodman and Said. Their basis functions reduce to the univariate Said-Ball basis functions on each side of the triangle. Since Wang-Ball curves are much better than Bézier curves and Said-Ball curves in recursive evaluation and degree elevation/reduction, it is worthwhile to develop bivariate Wang-Ball basis functions for free-form surface design.

The paper is organized as follows. Section 2 gives the new triangular Ball basis functions and their properties. Section 3 investigates some properties of triangular Wang-Ball surfaces, specially including degree elevation and recursive evaluation. A summary conclusion is given in Section 4.

## 2 Triangular Ball Basis

We now consider the construction of Wang-Ball basis functions on a triangle. We use control nets which are similar to Bézier nets of triangular Bézier patches, i.e., the nets are represented as \( \{ P_{i+j+k} \}, \) \( i + j + k = n \) for degree \( n. \) Then the new triangular Ball surfaces can be defined by the use of Wang-Ball basis \( B_{i,j,k}^n(u,v,w) \) and control points \( P_{i,j,k}, \) i.e.

\[ P(u,v,w) = \sum_{i+j+k=n} P_{i,j,k} B_{i,j,k}^n(u,v,w) \quad u,v,w \geq 0, \quad u + v + w = 1 \quad (2) \]

As stated in the previous section, Wang-Ball curves are much better than Bézier curves and Said-Ball curves in evaluation and degree elevation/reduction. Naturally, we expect that there exists similar construction for bivariate surface form. So we construct Wang-Ball basis \( B_{i,j,k}^n(u,v,w) \) such that

- \( B_{i,j,k}^n(u,v,w) \geq 0 \) and \( \sum_{i+j+k=n} B_{i,j,k}^n(u,v,w) = 1; \)
- The basis functions reduce to the univariate Wang-Ball basis functions if one of the three parameters \( u, v, \) \( w \) is zero.