An Adjoint Variable Method for Structural Design Sensitivity Analysis of a Distinct Eigenvalue Problem

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New adjoint variable method for design sensitivity analysis of distinct eigenvalues and eigenvectors is presented. In the viewpoint of efficiency for the design sensitivity analysis of eigenvectors especially, the developed adjoint variable method is required to compute adjoint variables from simultaneous linear equations, the so-called adjoint equations, instead of linear combination of eigenvectors. Once we obtain the adjoint variables, design sensitivity analysis of response function that is given in terms of eigenvalues, eigenvectors and design variables can be computed directly. In this way, design sensitivity analysis of eigenvectors can be obtained by using eigenvalues and their corresponding eigenvectors of the mode being differentiated only. To verify the proposed method, numerical examples are demonstrated. This can have considerable impact on computer implementation of the developed method in the design sensitivity analysis of eigenproblem needed for practical applications.

Key Words: Design Sensitivity Analysis, Adjoint Variable Method, Design Optimization, Eigenproblem, Modal Analysis

1. Introduction

Eigenproblems are commonly considered in structural stability, buckling, noise and vibration analyses. Design sensitivity analysis of an eigenproblem computes the rate of changes of response-dependent function, for instance, eigenvalues and eigenvectors, with respect to the perturbation of design variables. Design sensitivity analysis is also an essential step to systematically improve the existing design and to optimize a system with the aid of gradient-based optimization techniques (Hafika and Adelman 1989; Haug, et al 1986).

Fox and Kapoor (1968) developed general technique to compute design sensitivity of eigenvalues for symmetric matrices. However this method requires all eigenvalues and eigenvectors for the system, which is computationally expensive for a large-scale problems. Plaut and Husysin (1973) and Rudisill (1974) developed formulas for second-order design sensitivity analysis of eigenvalues.

To analyze the design sensitivity of the eigenvectors, we can use direct differentiation method that differentiates an eigenproblem with respect to design variables and solves the simultaneous equation directly for the eigenvector derivatives (Haug, et al 1986). However, since the simultaneous equation is singular, an orthonormality condition is adapted in the solution process (Jung, et al 1997; Lee and Jung 1997). Nelson introduced a normalization condition where the largest component of the eigenvector is unity (Nelson 1976). Note that Nelson and Lee and Jung methods require only the eigenvalues and eigenvectors for the modes being differentiated.

Design sensitivity analysis of eigenvectors can also be obtained by the modal method (Fox and Kapoor 1968; Rogers 1970) and a modified modal method (Wang 1991; Lin, Lim and Wang 1997), whereby the design derivatives of the eigenvectors are expanded in terms of the eigenvectors. The modal method approximates the derivatives of eigenvectors as a linear combi-
nation of eigenvectors. This method can be computationally expensive and impractical if large number of eigenvectors is needed to accurately represent the derivatives of eigenvectors. The modified modal method is developed to reduce the number of eigenvectors needed to represent the derivatives by including an additional term in the linear combination of eigenvectors, where inaccuracy of the approximation has been leaded for need of relatively higher modes. Review and comparison of several methods for design sensitivity analysis of eigenvectors have been carried out (Stulter, et al 1988). Note that the modal methods compute the design sensitivity of state variables instead of the design sensitivity of response functions.

Proposed new method for calculating derivatives of eigenvalues and eigenvectors is purely adjoint variable method that requires evaluation of the adjoint variables from the simultaneous system equation, the so-called adjoint equation. It is very important to note that the adjoint equation requires only the eigenvalues and associated eigenvectors of the modes being differentiated. Once we obtain adjoint variables, we can evaluate design sensitivity coefficients of response function directly. Therefore, when the dimensions of design variables are larger than the number of response functions, the developed method is generally more efficient than the direct differentiation method and the modal methods. Numerical examples are given to verify the developed method. Further, the developed method can be easily implemented into a commercial finite element program to carry out the design sensitivity analysis of eigenproblems needed for practical applications.

2. Definition of Eigenproblem

Undamped free vibration and linear buckling analysis lead to the generalized eigenproblem as follows:

\[ K u_i = \lambda_i M u_i \]  

where K represents the stiffness matrix, M represents the mass matrix in vibration analysis or geometric stiffness matrix in buckling analysis. The eigenvalue \( \lambda_i \) and associated eigenvectors \( u_i \) represent the i-th free vibration frequency squared and corresponding mode shape vectors, respectively. In case of buckling problems, the lowest eigenvalue \( \lambda_i = \lambda_{cr} \) is associated with buckling load.

Since the mode shape is often normalized with a symmetric positive definite matrix, we take M-orthonormality condition as follows:

\[ u_i^T M u_j = \delta_{ij} \]  

where \( \delta_{ij} \) represents the Kronecker delta and the right superscript T denotes the transposition of a matrix.

3. Design Sensitivity Analysis of Eigenproblem

Consider a general response function of an eigenproblem represented in terms of eigenvalues, eigenvectors, and a design variable as follows:

\[ g = g(\lambda_i, u_i, b) \]  

where \( b \) denotes design variable. It is assumed that the response function whose design sensitivity needs to be evaluated is continuous and differentiable with respect to its arguments.

To develop the adjoint variable method with the aid of the variational principle for design sensitivity analysis (Arora and Cardoso 1992), we first define the augmented function as

\[ A = g(\lambda_i, u_i, b) + z^T (K - \lambda_i M) u_i + y \left( \frac{1}{2} - \frac{1}{2} u_i^T M u_i \right) \]  

where \( y \) and \( z \) are the adjoint variables, the so-called Lagrange multipliers, for eigenvalues and eigenvectors, respectively, which will be determined later.

According to the variational principle for design sensitivity analysis, the total design variation of the response function can be represented as explicit design variation of the augmented function given in Eq. (4), i.e.,

\[ \frac{dg}{db} = \frac{\partial A}{\partial b} - \frac{\partial g}{\partial b} + z^T \left( \frac{\partial K}{\partial b} - \lambda_i \frac{\partial M}{\partial b} \right) u_i \]