Partial Completion of Equational Theories

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Received November 30, 1998; revised February 21, 2000.

Abstract In this paper, the notion of partial completion of equational theories is proposed, which is a procedure to construct a confluent term rewriting system from an equational theory without requirement of termination condition. A partial completion algorithm is presented with a brief description of its application in a program development system.

Keywords term rewriting system, completion of equational theory, partial completion

1 Introduction

Term rewriting systems (abbreviated as TRS) are a general computation model and play an important role in computer science. Rewriting techniques, such as the standard completion procedure (namely, the Knuth-Bendix completion procedure\cite{1-3}), have been applied to a variety of problems including word problems in universal algebra and proof of inductive properties of abstract data types. Completion is a procedure that attempts to generate a canonical (confluent and terminating) TRS for a given set of equations. The validity of the transformation from equational theory to TRS is assured by the fact that any two terms are equivalent in the equational theory if and only if they can be reduced into identical normal forms in the TRS\cite{4,5}. Many difficult problems can be solved using completion procedure\cite{6-8}.

Many researchers have attempted to generalize the standard completion procedure and proposed the notion of completion modulo equations\cite{9-12}. But there are limitations to these methods since they demand complicated conditions, such as termination of rewriting modulo equations, existence of the CSU (Complete Set of E-Unifiers), finiteness of the E-congruence class or well-founded subterm ordering modulo equations, and so on. They can be applied to AC-completion and several other cases, however, a lot of problems cannot be solved by this method, like the following example.

Example 1. 
\[(s \cdot x) \cdot y \cdot z = (x \cdot z) \cdot (y \cdot z),\]
\[(k \cdot x) \cdot y = x,\]
\[E(s, k) = false,\]
\[E(k, s) = false,\]
\[E(x, x) = true.\]

In fact, it is not always necessary to transform an equational theory into a canonical (i.e., confluent and terminating) TRS in solving many problems. Sometimes a confluent TRS instead of a canonical one is enough.

In this paper, we introduce the notion of partial completion. Informally, partial completion generates a confluent TRS $R_1 \cup R_2$, which is equivalent to a given equational theory $E$ with respect to a reduction ordering and a set $T$ of terms, where $R_2$ is a canonical TRS and $M \downarrow R_2$ is a normal form of $R_1$ for any $M \in T$. In particular, if $E$ can be completed by the Knuth-Bendix procedure, then $R_1 = \emptyset$, so partial completion is obviously more general than the standard completion. Let $R_1 \cup R_2$ be the result of the partial completion of $A$ with res-
pect to \(\{M, N\}\), then \(A \vdash M = N\) iff \(M \rightarrow R_2 = N \rightarrow R_2\).

Most of the various methods for deciding confluence are applied to terminating TRS only. Since partial completion does not require the termination condition of TRS, a method of checking the confluence property of a TRS without termination is needed. There has been no satisfying result in this area so far [213–19]. [20–24] proposed an efficient method of showing confluence of TRS without termination. An important concept of structure measure of TRS is defined in [24], and based on this notion, relatively pseudo-linear TRS is introduced, which can cover almost all the TRSs found in research work. A main theorem is proved: if \(R_1\) is a pseudo-linear TRS relative to \(R_2\), then the confluence of \(R_2\) implies the confluence of \(R_1 \cup R_2\). As a result of the application of this theorem, a partial completion algorithm is proposed in this paper.

For details of the basic concepts and notations in this paper, readers are referred to [2, 20, 21, 24, 26].

2 Relatively Pseudo-Linear TRS and Its Confluence

Let \(R\) be a TRS on set \(F\) of function symbols, \(l(R)\) and \(p(R)\) denote the sets of left-linear and non-left-linear rewrite rules of \(R\) respectively, \(RT(R) = \{l|l \rightarrow r \in R\}\), and \(pp(R) = RT(p(R))\). For any term \(M\), let

\[
NV(M) = \{x \in V(M) | \exists w_1, w_2 \in O(M) : M/w_1 = M/w_2 = x \wedge w_1 \neq w_2\},
\]

i.e., \(NV(M)\) is the set of variables in \(M\) which occurs more than once.

The structure measure introduced in [24] is very important for showing the confluence of term rewriting systems without termination. Based on the structure measure, we propose the notion of relatively pseudo-linear TRS and its properties.

**Definition 1.** Let \(\geq\) be a well-founded quasi-ordering on \(T(F)\). \(\geq\) is called a structure measure of \(R\) if and only if \(\geq\) satisfies

(i) \(M \rightarrow R N\) implies \(M \geq N\),

(ii) \(f(M_1, A, ..., M_n) \geq M_i\) for any \(i, 1 \leq i \leq n\) and \(f \in F_n, n > 0\),

(iii) \(\theta(M) > \theta(M)/w\) for any \(M \in pp(R)\), strong substitution \(\theta\) on \(M\), \(x \in NV(M)\) and \(w \in \text{occ}(M, x)\).

If \(\geq\) is a structure measure of \(R\) and \(\geq\) is monotonic and stable, then \(\geq\) is called a reduction measure.

In general, structure measure can be defined by the function interpretation method. A quick approach to defining structure measure is given in [25].

**Definition 2.** TRS \(R_1\) is called a pseudo-linear TRS relative to \(R_2\) if and only if the following conditions hold.

(i) \(R_2\) is a left-linear TRS.

(ii) There exists a structure measure on \(R_1 \cup R_2\).

(iii) For any \(l_1 \rightarrow r_1 \in p(R_1), l_2 \rightarrow r_2 \in R_1 \cup R_2\) and \(w \in O(l_a, a \in \{1, 2\}, if l_a/w is not unifiable with \(l_b, b \in \{1, 2\} \setminus \{a\}\), then there are no strong substitution \(\theta_a\) to \(l_a/w\) and strong substitution \(\theta_b\) to \(l_b\) such that \(\theta_a(l_a/w) = \theta_b(l_b)\). This property is denoted as \(NS(l_1 \rightarrow r_1, l_2 \rightarrow r_2)\).

(iv) For any \(l_1 \rightarrow r_1 \in p(R_1), l_2 \rightarrow r_2 \in R_1 \cup R_2\), if \(l_1 \rightarrow r_1\) and \(l_2 \rightarrow r_2\) are overlapping, and suppose \(\theta\) is the most general unifier (mgu) of \(l_a/w\) and \(l_b\) (where \(a, b = \{1, 2\}\)) in constructing a critical pair \((P, Q)\), then for any \(x \in NV(l_1) \cup NV(l_2)\), and any \(M \in RT(R)\), where \(\theta(x)\) and \(M\) are not strongly overlapping, there exists \(N\) such that \(P \rightarrow^{*} N\) and \(Q \rightarrow^{*} N\), and for any one-step reduction \(M_1 \rightarrow^{*} M_2\) in \(P \rightarrow N\) or \(Q \rightarrow N\) and any substitution \(\eta\), there is \(i \in \{j | l_j \rightarrow r_j \in R \land 1 \leq j \leq 2\}\) such that \(\eta(\theta(l_i)) > \eta(M_1/w')\).

**Example 2.** Consider Example 1 above, and let \(R_1\) and \(R_2\) be as follows: