AN APPROXIMATE METHOD TO COMPUTE THE NONLINEAR NORMAL MODES AND BIFURCATION BY THE PRINCIPLE OF LEAST ACTION

Chol Hui Pak* and Young Suk Yun*

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An analytical procedure is presented to find approximately the nonlinear normal modes in conservative two-degree-of-freedom system by using the principle of least action and by assuming that the modal curve is straight. The results are compared with those of numerical experiments by utilizing the 4th order Runge-Kutta method, and it is found that there are good agreements between them. By utilizing this procedure, it is demonstrated to compute the normal modes which are analytically extended from the linearized modes and to find the generically or non-generically bifurcated modes which do not have any counterpart in the linear theory.

Key Words: Nonlinear Normal Mode, Generic Bifurcation, Non-Generic Bifurcation, Modal Curve, Homogeneous System, Principle of Least Action

1. INTRODUCTION

We shall be interested in formulating an approximate procedure to compute the normal modes in nonlinear conservative systems having two degrees of freedom. As usual in treating nonlinear vibrations in terms of normal mode, we shall be concerned with motions of large amplitudes, the total number of modes, and the bifurcation phenomenon.

The existence of normal modes has been extensively investigated by (Pak and Rosenberg, 1966). Yen (1974) demonstrated that the normal modes occur in pairs. Johnson and Rand (1979) showed that the normal modes generically occur in pairs, and the non-generic case corresponds to the bifurcation.

The procedure to compute the normal modes and the bifurcation in general nonlinear systems has not been reported. For a special type of nonlinear systems called a symmetric system, Anand (1972) computed the normal modes and the bifurcation. The method is not applicable to other nonlinear systems. The method is based on the assumption that the modal curves of both similar and nonsimilar normal modes are straight. The modal curves of nonsimilar modes computed by the fourth order Runge-Kutta method are found to be approximately straight (Pak and Park, 1988).

The purpose of this paper is to formulate a procedure to approximately find the normal modes by utilizing the principle of least action under the assumption that the modal curves are straight. The total number of normal modes, which a system can possess, will be determined through this procedure.

Examples are illustrated for systems having cubic nonlinearity. The locations of normal modes are calculated and the global distribution of normal modes due to the increase of total energy of the system is also demonstrated. The results obtained by present procedure are compared with computer solutions.

2. PRELIMINARIES

2.1 Nonlinear Two-Degree-of-Freedom System

Let us consider the nonlinear conservative two degrees of freedom system. The mathematical model of the system consists of two concentrated masses \( m_1 \) and \( m_2 \), connected to each other by means of massless coupled spring and to the wall of each side by means of massless anchor spring, as shown in Fig. 1.

The spring forces of this system are characterized by odd order polynomial

\[
G_i = k_i J + a_i J^3 (i = 1, 2, 3)
\]

where \( J \) is elongation of spring beyond its unstretched length. In this paper, the above nonlinear, conservative, two degrees of freedom system is called the system \( S \). And the system is called symmetric system if

\[
m_1 = m_2 = m, \quad a_1 = a_2 = a, \quad \text{and} \quad k_1 = k_3 = k,
\]

otherwise, it is called unsymmetric system.

[Fig. 1 Non-linear two-degree-of-freedom system]
2.2 Definition of Normal Mode
The nonlinear normal mode of two-degree-of-freedom system is defined as follows:

1. $x_1(t) = x_1(t + \tau)$; every mass does the same periodic motion.
2. $x_1(t_0) = x_1(t_0) = 0$; every mass comes to the equilibrium state simultaneously.
3. $\dot{x}_1(t) = \dot{x}_1(t_0) = 0$; every mass has its extreme value at the same time $t_0$.
4. $x_2 = x_2(x_1)$; $x_2$ is a single-valued function of $x_1$ in the closed domain $V(x_1, x_2) = h$ of the $x_1, x_2$-plane.

These normal modes have very important meaning in the view that the resonance occurs when the frequency of oscillatory force lies close to the natural frequencies and, in the neighborhood of resonance, linear or not, is subjected to oscillatory forces (Rosenberg, 1966).

The normal modes can be depicted in configuration space as shown in Fig. 2. The trajectory is called modal curve. If the modal curve is straight, it is called the similar normal mode, otherwise it is called nonsimilar one (Rosenberg, 1960, 1961).

2.3 Equations of Motion
The kinetic energy $T$ and the potential energy $V$ of the system are in the form

$$T = \frac{1}{2}(m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2)$$

$$V = V_1 + V_2$$

where

$$V_1 = \frac{1}{2} \left\{k_1 x_1^2 + k_2 (x_2 - x_1)^2 + k_3 x_1 x_2 \right\}$$

$$V_2 = \frac{1}{4} \left\{a_1 x_1^4 + a_2 (x_2 - x_1)^4 + a_3 x_1^3 \right\}$$

The equations of motion can be written in the form:

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 + a_1 x_1^3 - a_2 (x_2 - x_1)^2 = 0$$

$$m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_3 x_1 + a_2 x_2^3 + a_3 x_2 (x_2 - x_1)^2 = 0$$

By means of the coordinate transformations

$$x = \sqrt{m_1} x_1$$

and

$$y = \sqrt{m_2} x_2$$

the kinetic energy $T$ and the potential energy $V$ can be rewritten as

$$T = \frac{1}{2}(x^2 + y^2)$$

$$V = \frac{1}{2} \left\{ \frac{k_1}{m_1} x^2 + k_2 \left( \frac{y}{\sqrt{m_2}} - \frac{x}{\sqrt{m_1}} \right)^2 + \frac{k_3}{m_2} y^2 \right\}$$

$$+ \frac{1}{4} \left\{ \frac{a_1}{m_1} x^4 + a_2 \left( \frac{y}{\sqrt{m_2}} - \frac{x}{\sqrt{m_1}} \right)^4 + \frac{a_3}{m_2} y^4 \right\}$$

Then the equations of motion can be written

$$\ddot{x} = -\frac{\partial}{\partial x} V\left(\frac{x}{\sqrt{m_1}}, \frac{y}{\sqrt{m_2}}\right)$$

$$\ddot{y} = -\frac{\partial}{\partial y} V\left(\frac{x}{\sqrt{m_1}}, \frac{y}{\sqrt{m_2}}\right)$$

Equation (7) are mathematically completely equivalent to equations (4); however their physical meanings are quite different. Equation (7) may be regarded as the equations of motion of mass points having unit mass that move in the $xy$-plane.

When moving in the configuration space, the unit mass point traces out a trajectory. Here we can derive the equation of this trajectory. The Eq. (7) can be transformed into the equation of trajectory of mass point in the configuration space as follows (Rosenberg, 1966):

$$2[k - V(x, y)]\dot{y}^2 + (1 + y'^2)\left( V_y - y' V_x \right) = 0$$

where $h$ is the total energy of the system.

3. BASIC THEORY
The system considered in this paper is a holonomic and conservative system. Therefore, the periodic motions of the system can be calculated by applying the Jacobi’s principle of least action (Meirovitch, 1970 and Rosenberg, 1977). In holonomic systems possessing an energy integral in which the energy level is fixed in moving from a prescribed initial configuration $P_1$ to a prescribed terminal configuration $P_2$, the action integral

$$A = \int_{P_1}^{P_2} \sqrt{2(h - V)} \, ds,$$

where $ds = \sqrt{dx^2 + dy^2}$, along the actual trajectory connecting initial and terminal configurations is stationary relative to all other trajectories connecting the same end configurations, that is,

$$\delta A = 0,$$

Thus the principle of least action gives necessary and sufficient conditions for the action $A$ to have a stationary value for the actual motion of the system. Therefore, the normal mode is found by obtaining the point $P$ which make the action $A$ stationary when $P_1 = 0$ and $P_2 = P$. When $ds = \sqrt{\dot{x}^2 + \dot{y}^2} dt$, the principle becomes Euler-Lagrange Eq. (7) and when $ds = \sqrt{1 + y'^2} \, dx$, the principle results in the equation of trajectory (8).

However, it is very difficult to solve the Eq.(8), since the Eq. (8) is intrinsically nonlinear, even when the springs are linear, and coefficient of $y''$ can be zero due to the definition of normal modes. But if the normal modes are similar, that is,