**On Stochastically Perturbed Differential Systems.**

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**SUNTO** - Si dimostra un risultato relativo all’esistenza globale e alla unicità traiettoria per traiettoria delle soluzioni del sistema differenziale stocastico di Itô. Le condizioni sufficienti che vengono proposte migliorano dei risultati di Taniguchi [5] e Constantin [1].

**ABSTRACT** - We give a result which refers to the global existence and pathwise uniqueness of solutions of Itô stochastic differential systems. The sufficient conditions we propose improve some results of Taniguchi [5] and Constantin [1].

1. - Introduction.

Let \((\Omega, \mathcal{A}, P)\) be a complete probability space and let \(\{\mathcal{A}_t, t \geq 0\}\) be a family of complete \(\sigma\)-subalgebras of \(\mathcal{A}\) satisfying \(\mathcal{A}_s \subset \mathcal{A}_t\) if \(0 \leq s < t\). Let \(W(t), t \geq 0\) be a \(\mathbb{R}^m\)-valued Brownian motion defined on \((\Omega, \mathcal{A}, P)\) with respect to the family \(\{\mathcal{A}_t, t \geq 0\}\).

We consider the Itô stochastic differential system with initial condition

\[
\begin{align*}
(1) \quad dX(t) &= F(t, X(t)) \, dt + G(t, X(t)) \, dW(t), \\
(2) \quad X(t_0) &= X_0,
\end{align*}
\]

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with \( F \in \mathcal{C}(R_+ \times R^n, R^n) \) and \( G \in \mathcal{C}(R_+ \times R^n, R^m) \) and \( X_0 \) a \( \mathcal{C}_0 \)-measurable \( R^n \) valued function, independent of the Brownian motion \( W(t), t \geq t_0 \), and which verifies \( E\|X_0\|^2 < \infty \).

A solution \( X(t) \) of (1) is a random process defined for \( t \in [t_0, t_0 + \varepsilon] \), such that \( X(t) \) is \( \mathcal{C}_0 \)-measurable, is sample continuous, for some \( \varepsilon > 0 \) verifies

\[
\int_{t_0}^{t_0 + \varepsilon} (\|F(s, X(s))\|^2 + \|G((s, X(s))\|^2) \, ds < \infty \quad a.e.
\]

and satisfies (1). By uniqueness we mean pathwise uniqueness, i.e. if \( X_1(t) \) and \( X_2(t) \) are two solutions of (1) defined on \([t_0, t_0 + \varepsilon]\) and satisfying the same initial condition (2), then

\[
P\left( \sup_{t_0 \leq t \leq t_0 + \varepsilon} \{|X_1(t) - X_2(t)|\} = 0 \right) = 1.
\]

There exist many results related with the existence and the uniqueness of the solutions of (1). The classical sufficient conditions proposed by Itô were improved by Yamada [7]. Taniguchi [5] extended the results of Yamada proposing the following sufficient conditions for existence and uniqueness:

\[
(T1) \quad \|F(\cdot, 0)\| \in L^2_{\text{loc}}(R_+, R_+) \quad \text{and} \quad \|G(\cdot, 0)\| \in L^2_{\text{loc}}(R_+, R_+);
\]

\[
(T2) \quad \|F(t, x) - F(t, y)\|^2 + \|G(t, x) - G(t, y)\|^2 \leq b(t) f(\|x - y\|^2),
\]

for all \( x, y \in R^n \) where \( b \in \mathcal{C}(R_+, R_+) \) and \( f \in \mathcal{C}(R_+, R_+) \) is a monotone non-decreasing, concave function which satisfy \( f(0) = 0 \) and

\[
\lim_{r \to 0} \int_0^1 \frac{ds}{f(s)} = \infty.
\]

Although this result is very general, in perturbation problem is it is important to have separate conditions on the terms \( F \) and \( G \). Especially interesting is the following problem. Let us take the ordinary differential equation

\[
x' = F(t, x)
\]

with \( F \) satisfying (T1) and (T2) (so that we have uniqueness of solutions) and consider the stochastically perturbed equation (1). Under what conditions on \( G \) are have pathwise uniqueness for the solutions of (1)?