A MULTIPOINT PROBLEM WITH MULTIPLE NODES FOR LINEAR HYPERBOLIC EQUATIONS

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We establish conditions for the unique solvability of a multipoint (with respect to the time coordinate) problem with multiple nodes for linear hyperbolic equations with constant coefficients in the class of functions periodic in the space variable. We prove metric statements concerning lower bounds of small denominators that appear in the course of construction of a solution of the problem.

1. Problems involving partial differential equations with multipoint conditions with respect to a selected variable are, generally speaking, conditionally well posed, and, in numerous cases, their solvability is connected with the problem of small denominators. For the case of simple nodes, such problems were studied in numerous works (see, e.g., [1-12]).

In the present paper, which is a sequel of [3], we investigate a multipoint problem with multiple nodes for \( n \)-th order \( (n > 2) \) hyperbolic equations with constant coefficients in the class of functions \( 2\pi \)-periodic in the space variable. An analogous problem for one class of differential-operator equations was studied in [13].

We use the following notation: \( \Omega_{2\pi}^1 \) is a one-dimensional torus, i.e., a circle of the unit radius, \( D^1 = \{ (t, x) \in \mathbb{R}^2 : t \in [0, T], x \in \Omega_{2\pi}^1 \} \), and \( \Gamma \) is the space of trigonometric polynomials

\[
P_m(x) = \sum_{k=-m}^{m} C_k \exp(ikx), \quad x \in [0, 2\pi], \quad m = 0, 1, \ldots,
\]

with complex coefficients in which the convergence is defined as follows: \( \Gamma \ni P_n \rightarrow P \) if the powers of all polynomials \( P_n(x) \) do not exceed a certain fixed number \( N \) and \( P_n(x) \rightarrow P(x) \) as \( n \rightarrow \infty \). Further, \( \Gamma' \) is the space of all linear continuous functionals over \( \Gamma \) with weak convergence, which coincides with the space of formal trigonometric series [14]. \( H_{\alpha}(\Omega_{2\pi}^1), \alpha \in \mathbb{R}, \) is the Hilbert space of complex-valued \( 2\pi \)-periodic functions of the form

\[
y(x) = \sum_{|k| \geq 0} y_k \exp(ikx)
\]

with the norm

\[
\|y(x)\|_{H_{\alpha}(\Omega_{2\pi}^1)}^2 = 2\pi \sum_{|k| \geq 0} (1 + k^2)^\alpha |y_k|^2,
\]
$H_\alpha^n(D^1), \alpha \in \mathbb{R}, n \in \mathbb{Z}_+, \alpha \leq n$, is the Hilbert space of functions $h(t, x)$ such that, for each $t \in [0, T]$, $\partial^s h/\partial t^s \in H_{\alpha-s}(\Omega^n_{2\pi})$, $s = 0, 1, \ldots, n$, and

$$\|h(t, x)\|^2_{H_\alpha^n(\Omega^n_{2\pi})} = \int_0^T \sum_{s=0}^n \|\partial^s h(t, x)\|^2_{H_{\alpha-s}(\Omega^n_{2\pi})} dt < \infty,$$

$C^n([0, T], \Gamma)$ ($C^n([0, T], \Gamma')$) is the class of functions $g(t, x)$ such that, for arbitrary $t \in [0, T]$, $\partial^j g/\partial t^j \in \Gamma(\Gamma'), j = 0, 1, \ldots, n$.

2. In the domain $D^1$, we consider the problem

$$L(u) = \sum_{s=0}^n a_s \partial^s u(t, x) = 0, \quad a_s \in \mathbb{R}, \quad a_n = 1, \quad (1)$$

$$N_{jm_j}(u) = u^{(m_j-1)}(t_j, x) = \Phi_{jm_j}(x) \left\{ \sum_{i=1}^j r_i = n, \quad 0 \leq t_1 < t_2 < \ldots < t_j \leq T \right\}, \quad (2)$$

where $2 \leq r_j \leq n - 1$, $j = 1, \ldots, l$. Equation (1) is strictly hyperbolic in the Petrovskii sense, i.e., all roots $\lambda_j, j = 1, \ldots, n$, of the equation

$$\sum_{s=0}^n a_s \lambda^s = 0$$

are real and different. The form of the domain $D^1$ imposes conditions of $2\pi$-periodicity in the variable $x$ on the functions $u(t, x)$ and $\Phi_{jm_j}(x), m_j = 1, \ldots, r_j, j = 1, \ldots, l$.

We seek a solution of problem (1), (2) in the form of a series

$$u(t, x) = \sum_{|k| \geq 0} u_k(t)\exp(ikx). \quad (3)$$

Each function $u_k(t), k \in \mathbb{Z}$, is determined as a solution of the following multipoint problem for an ordinary differential equation:

$$\sum_{s=0}^n a_s (ik)^{n-s} u_k^{(s)}(t) = 0, \quad (4)$$

$$u_k^{(m_j-1)}(t_j) = \Phi_{jm_j}(k), \quad m_j = 1, \ldots, r_j, \quad j = 1, \ldots, l, \quad (5)$$

where

$$\Phi_{jm_j}(k) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_{jm_j}(x) \exp(-ikx) dx, \quad k \in \mathbb{Z}. \quad (6)$$