ON THE BEST APPROXIMATION OF FUNCTIONS OF \( n \) VARIABLES

N. P. Korneichuk

We propose a new approach to the solution of the problem of the best approximation by a certain subspace for functions of \( n \) variables determined by restrictions imposed on the modulus of continuity of certain partial derivatives. This approach is based on the duality theorem and on the representation of a function as a countable sum of simple functions.

1. Introduction and the Duality Theorem

We do not present here the theory of the problem because it would take too much space; we only recall scientists whose results concerning the approximation in the multidimensional case are well known: Nikol’skii, Temlyakov, Bugrov, Potapov, Babenko, Galeev, Din Zung, et al.

In the present paper, we propose a new approach that enables us, in certain cases, to obtain exact results for periodic functions of \( n \) variables concerning the estimation of the best approximation by a subspace. We start from the following statement, which is called the duality theorem for the best approximation [1] (see also, e.g., [2, p. 113]):

**Theorem A.** Suppose that \( X \) is a liner normed space, \( X^* \) is the space dual to \( X \), and \( F \) is a subspace of \( X \). For any \( x \in X \setminus F \), we have

\[
E(x, F) := \inf_{u \in F} \| x - u \| = \sup_{f(x) \in X^*} \{ f(x) : f \in X^*, \| f \| \leq 1, f(u) = 0 \ \forall u \in F \}.
\]

In Theorem A, let \( X \) be the space \( L_{n,p}, \ 1 \leq p \leq \infty \), of functions \( f(\bar{x}) = f(x_1, x_2, \ldots, x_n) \) \( 2\pi \)-periodic in each variable with the ordinary norm

\[
\| f \|_{L_{n,p}} = \| f \|_{L_{n,p}} = \begin{cases} \left( \frac{2\pi}{0} \int |f(\bar{x})|^p \, d\bar{x} \right)^{1/p}, & 1 \leq p < \infty; \\ \sup_{\bar{x}} |f(\bar{x})|, & p = \infty. \end{cases}
\]

Taking into account the general form of a linear functional in the space \( L_{n,p}, \ p \geq 1 \) (see, e.g., [3, p. 196]), and Proposition 1.4.2 in [4, p. 26], which is obviously true in the \( n \)-dimensional case as well, the statement of Theorem A, i.e., relation (1), can be represented for \( \bar{x}(t) \in L_{n,p} \setminus F \) as follows:

\[
E(x, F)_p = \sup \left\{ \int_0^{2\pi} x(\bar{r})h(\bar{r}) \, d\bar{r} : h \in L_{n,p'}, \| h \|_{p'} \leq 1, \ \int_0^{2\pi} u(\bar{r})h(\bar{r}) \, d\bar{r} = 0 \ \forall u(\bar{r}) \in F \right\},
\]

\[
1 \leq p < \infty, \ 1/p + 1/p' = 1.
\]

Note that, for \( p = \infty \), equality (2) is valid at least for finite-dimensional subspaces \( F \) [1; 4, p. 26].
2. Modulus of Continuity

In the one-dimensional case, the modulus of continuity \( \omega(f, \delta) \) of a function \( f(t) \in C[a, b] \) is defined by the relation

\[
\omega(f, \delta) = \sup \{ |f(t') - f(t'')| : t', t'' \in [a, b], |t' - t''| \leq \delta \},
\]

which, for an absolutely continuous function \( f(t) \), can be rewritten in the form

\[
\omega(f, \delta) = \sup \left\{ \left| \int_{t'}^{t''} f'(t) \, dt \right| : t', t'' \in [a, b], |t' - t''| \leq \delta \right\}.
\]

Condition (3), which defines the modulus of continuity, can be generalized to the case of a function \( f(x_1, x_2, \ldots, x_n) \) of \( n \) variables.

We specify the distance \( \rho(\overline{x}, \overline{y}) \) between the points \( \overline{x} = \{ x_1, x_2, \ldots, x_n \} \) and \( \overline{y} = \{ y_1, y_2, \ldots, y_n \} \) in \( \mathbb{R}^n \). This distance in \( \mathbb{R}^n \) determines the unit sphere \( B_\rho \) centered at the origin:

\[
B_\rho = \{ \overline{x} : \overline{x} \in \mathbb{R}^n, \ \rho(\overline{x}, \overline{0}) \leq 1 \}.
\]

By \( B_\rho(\overline{a}, r) \) we denote the ball of radius \( r \) centered at the point \( \overline{a} \in \mathbb{R}^n \) and determined by the distance \( \rho \), namely,

\[
B_\rho(\overline{a}, r) = \{ \overline{x} : \overline{x} \in \mathbb{R}^n, \ \rho(\overline{a}, \overline{x}) \leq r \}.
\]

If \( \overline{a} = \overline{0} \), then we write \( B_\rho(r) \) instead of \( B_\rho(\overline{0}, r) \).

The quantity

\[
\omega_\rho(f, \delta) = \sup \left\{ \left| \int_{B_\rho(\overline{a}, r) \cap \overline{Q}} f(\overline{x}) \, d\overline{x} \right| : \overline{a} \in \overline{Q}, \ \mes B_\rho(\overline{a}, r) \leq \delta \right\}
\]

(where the upper bound of the modulus of the integral is taken over all balls \( B_\rho(\overline{a}, r) \) centered at the point \( \overline{a} \in \overline{Q} \) whose measure does not exceed \( \delta \)) is called the modulus of continuity corresponding to the function \( f(\overline{x}) \) summable in a bounded closed domain \( \overline{Q} \subset \mathbb{R}^n \).

For a fixed distance \( \rho \), the modulus of continuity \( \omega_\rho(f, \delta) \) for any function \( f(\overline{x}) \in \overline{Q} \subset \mathbb{R}^n \) possesses the following properties:

(i) \( \omega_\rho(f, 0) = 0 \);

(ii) the function \( \omega_\rho(f, \delta) \) is continuous and nondecreasing for \( 0 \leq \delta \leq \mes \overline{Q} \);

(iii) \( \omega_\rho(f, \delta) \) is a semiadditive function on the same set.

Specifying the function \( \omega(\delta) \) with the same properties and the distance \( \rho \) in \( \mathbb{R}^n \), we define the class \( H_\rho^0 \) of functions \( f(\overline{x}) \) summable on the set \( \overline{Q} \) and such that

\[
\omega_\rho(f, \delta) \leq \omega(\delta), \quad \delta \in [0, \mes \overline{Q}].
\]