ON THE FUNCTION OF HAMILTONIAN ACTION FOR NONHOLONOMIC SYSTEMS AND ITS APPLICATION TO THE INVESTIGATION OF STABILITY

S. P. Sosnitskii

For nonholonomic systems, we introduce the notion of the function of Hamiltonian action, with the use of which we investigate the stability of nonholonomic systems in the case where the equilibrium state under consideration is a critical point of the corresponding Lagrangian (Whittaker system).

As is known [1], unlike holonomic systems, equilibrium states of nonholonomic systems do not necessarily coincide with the critical points of the corresponding Lagrangian. This may be a source of additional difficulties in the investigation of the stability of a system, especially in the case where the problem is not solvable within the framework of the linear approximation. The stability of the equilibrium states of nonholonomic systems that are critical points of the Lagrangian was studied by Whittaker [2]. This narrower class of nonholonomic systems (which are called the Whittaker systems) is interesting, first of all, because its properties are rather similar to the properties of holonomic systems. Moreover, in a certain sense, one can consider the Whittaker system as a special kind of perturbed holonomic system, keeping in mind the possibility of applying the methods for the investigation of the stability of holonomic systems [3–6]. It also turns out that, for these systems, by analogy with holonomic ones, consideration of the function of Hamiltonian action $S$ is reasonable. In particular, within the framework of the approach of [7], which is based on the use of the function of action for the investigation of the stability of holonomic systems, it is possible to establish conditions of instability of equilibrium states for Whittaker systems.

1. Consider a nonholonomic system

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = B^T(q) \lambda, \quad \lambda = (\lambda_1, \ldots, \lambda_l)^T, \quad (1)$$

$$B(q) \frac{dq}{dt} = 0, \quad (2)$$

where $B(q) = (b_{ij}(q))$ is an $l \times n$ matrix, $i = 1, \ldots, l$, $j = 1, \ldots, n$, $l < n$, $\lambda$ is the set of constraints, $L(q, \dot{q})$, $B(q) \in C^2(D_q \times R^n_q)$, and the Lagrangian $L$ is given by the expression

$$L(q, \dot{q}) = T(q, \dot{q}) - \Pi(q) = \frac{1}{2} q^T A(q) q - \Pi(q). \quad (3)$$

Here, the functions $T$ and $\Pi$ are, respectively, the kinetic and potential energy of the system. We assume that the quadratic form $T(0, \dot{q})$ is positive definite, $\Pi(0) = 0$; $\partial \Pi / \partial q(0) = 0$ and, hence, the point $q = \dot{q} = 0$ is an equilibrium state of system (1)–(3).

The nonintegrable relations (2), which restrict the generalized velocities of the system (rank $B(q) = l$), are nonholonomic constraints.

As in the case of holonomic systems, for systems (1)–(3), we have the integral of energy
\[ T(q, \dot{q}) + \Pi(q) = h = \text{const.} \]  

It is known [8, 9] that the equations of motion of a nonholonomic system can be obtained on the basis of the Hamilton principle in the Hölder form

\[ \int_0^t \delta L(q, \dot{q}) \, d\tau = 0. \]  

Unlike the case of holonomic systems, the Hamilton principle in the form (5) is not the principle of stationary action in the case where the equality

\[ \int_0^t \delta L(q, \dot{q}) \, d\tau = 0 \]  

is satisfied. Apparently, Hertz [10] was the first to claim that the Hamilton principle in the form (6) is not true for nonholonomic systems. However, we note the following important fact: for the case of nonholonomic systems, the Lagrangian \( L(q, \dot{q}) \) also remains the key characteristic of a system.

On the basis of this fact, we consider the function

\[ S = \int_0^t L(q, \dot{q}) \, d\tau, \]  

where the quantities

\[ q = q(t, q_0, \dot{q}_0), \quad \dot{q} = \dot{q}(t, q_0, \dot{q}_0), \]  

\[ q_0 = q(t = 0), \quad \dot{q}_0 = \dot{q}(t = 0), \]  

that enter the integrand of equality (7) are the general solution of Eqs. (1), (2). Similarly to the case of holonomic systems, we call the function \( S \) the function of Hamiltonian action.

Assume that solution (8) is extendable to the whole axis \( t \in \mathbb{R} \) and, therefore, corresponds to the definition of flow [11]. This fact does not restrict the generality of consideration, since, in what follows, we consider the instability of equilibrium. Replacing first \( t \) by \( \tau \) in (8) and performing integration in equality (7), we obtain

\[ S = \tilde{S}(\tau, q_0, \dot{q}_0) \mid_{0}^{t} \in C_{q_0, q_0}^{(1,1)}(R \times s_{\delta}), \]  

where the vector \((q_0, \dot{q}_0)\) belongs to the neighborhood \( s_{\delta} = \{(q_0, \dot{q}_0) \in D_q \times R^n_q, \|q_0 \oplus \dot{q}_0\| < \delta\} \) of the point \( q = \dot{q} = 0 \). Taking into account that relations (8) define a flow so that

\[ q_0 = q(-t, q, \dot{q}), \quad \dot{q}_0 = \dot{q}(-t, q, \dot{q}), \]  

on the basis of (9) we have

\[ S = S^\tau(\tau, q(\tau), \dot{q}(\tau)) \mid_{0}^{t} \in C_{i q q}^{(1,1)}(R \times D_q \times R^n_q). \]  

In what follows, we represent system (1)–(3) in the form