mory for the Twenty-Fifth International Congress of Mathematicians.

I'm sure each individual reader would find many things of interest and many new ideas in Poincaré’s Prize. I will just mention a few that came my way, in order to suggest that there is indeed a genuine richness that lies between the covers of this book.

Much to my amazement I learned that the Poincaré dodecahedral space, which he discovered in 1904 as a counterexample to the “theorem” he had claimed in 1900, to this day remains the only known counterexample to this false theorem. I learned that RH were not the initials of that giant of topology, R H Bing, but were in fact his actual first name. More importantly, I also learned that Bing, who himself made prolonged serious attempts on the Poincaré Conjecture, in the end may have concluded that the conjecture was false. This, for me, was one of the very best moments in Szpiro’s book because it showed so dramatically how hard it is in mathematics to know what is true and what is false if someone of RH Bing’s stature can be so completely wrong about such a fundamental question as the Poincaré Conjecture.

From a more personal point of view, I was delighted to learn that James Alexander, who is well known to most mathematicians as the discoverer of the famous and fabulous “Alexander horned sphere,” has a classic technical climbing route up 14,255 foot-high Longs Peak in Colorado, named Alexander’s Chimney, after him. Similarly, I found a rather lengthy history of the École Polytechnique—which Poincaré attended from 1873 to 1875, graduating second in his class—absolutely fascinating because my wife directed a program at Colorado College in Colorado Springs during the mid-1980s. That program was designed for students from the École Polytechnique to guide them in the study of English and to acquaint them with American culture.

Here is one final nugget in the same vein that takes on special irony now that Perelman has rejected the Fields Medal: at his death in 1912, Poincaré had received the largest number of nominations for a Nobel Prize of any non-winner. Recall that there is not a mathematics category for Nobel Prizes and that it is for this reason that the Fields Medal is considered to be the mathematical equivalent of a Nobel Prize. Poincaré’s nominations were all in physics.

I must admit that, in the end, Szpiro does deliver on the tabloid promise from the back cover of his book. The most gripping, hard-to-put-down reading are Szpiro’s last two chapters, when he finally gets down to discussing the very messy controversy surrounding the solution of Poincaré’s famous conjecture. It is hard not to be intrigued by this controversy. There are some very serious issues here: What share of the credit for the solution does Hamilton deserve? That the Clay Institute may well award him a large share of the million-dollar prize is just one measure of the fact that many people believe that he and Perelman share equally in arriving at the final solution. Another of the serious issues is the way in which Perelman bypassed traditionally accepted methods for publishing mathematical proofs by placing his unrefereed proofs on the Internet. It has taken three years and several teams of heavily financed experts to conclude that Perelman’s work is correct. Meanwhile, the really messy part of the controversy arose from a claim made by a team of Chinese mathematicians that they had published the first complete proof of the Poincaré Conjecture. The article in The New Yorker greatly inflamed this controversy by including a full-page drawing with Shing-Tung Yau looking as if he is about to rip the Fields Medal from the neck of Perelman (who looks a bit like Vincent Van Gogh in this drawing).

Szpiro sorts through the details and complexities—including the ethical ones—of this controversy quite thoroughly and with what seems to be considerable fairness and a great deal of wisdom, both concerning human nature and with respect to maintaining always the highest regard for the well-being of mathematics. Thus, he is able to bring us beyond the controversy to the point where we can celebrate the solution of the Poincaré Conjecture, perhaps dream of solutions to one of the remaining millennium problems, and find other ways—in the words of the mission statement of the Clay Institute—“to further the beauty, power, and universality of mathematical thinking.”

REFERENCES

More math comics by Courtney Gibbons are available online at: brownsharpie.courtneygibbons.org

Department of Mathematics and Computer Science
Colorado College
Colorado Springs, CO 80903
USA
email: jwatkins@coloradocollege.edu

Fixing Frege
John Burgess

Reason’s Proper Study: Essays Towards a Neo-Fregean Philosophy of Mathematics
Bob Hale and Crispin Wright

REVIEWED BY Øystein Linnebo

We know that there are infinitely many prime numbers and that every natural number...
This makes sense because \( \approx \) is an equivalence relation.

The second step of Frege's defense of logicism provides an explicit definition of terms of the form \( \#F \). Frege does this in a theory that consists of second-order logic and his "Basic Law V," which states that the extension of a concept \( F \) is identical to that of a concept \( G \) if and only if the \( Fs \) and the \( Gs \) are co-extensional; or, in contemporary notation

\[
\forall x (Fx \leftrightarrow Gx).
\]

In this theory, Frege defines \( \#F \) as the extension of the concept "\( x \) is an extension of some concept equinumerous with \( F \)." That is, he defines

\[
\#F = \{x \mid \exists G (x = \{y \mid Gy \} \land F \approx G) \}.
\]

This definition is easily seen to satisfy (HP). More interestingly, Frege proves in meticulous technical detail how this definition and his theory of extensions entail all of ordinary arithmetic.

However, just as the second volume of his *magnum opus* was going to press in 1902, Frege received a letter from the English logician and philosopher Bertrand Russell, who reported that he had "encountered a difficulty" with Frege's theory of extensions. The difficulty Russell had encountered is the paradox now bearing his name. Frege's theory of extensions is in effect a naïve theory of sets. We may thus consider the set of all sets that are not members of themselves. In Frege's theory we can then prove that this set both is and is not an element of itself. Frege's response to Russell's letter is remarkable. Sixty years later Russell described it as follows.

As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure, clearly submerging any feelings of personal disappointment.

Russell's paradox eventually led Frege to give up on logicism. Until the 1980s both logicians and philosophers regarded Fregean logicism as a dead end, and people attracted to the idea of logicism pursued other versions of it, such as Russell's very complicated "type theory."

However, over the past two decades there has been a resurgence of interest in Fregean logicism. A variety of consistent fragments of Frege's theory have been identified and explored, and their possible philosophical significance has been vigorously debated. The two books under review are without doubt among the most important products of this resurgence. *Reason's Proper Study* is the most extensive philosophical articulation and defense to date of a specific neo-Fregean programme, whereas *Fixing Frege* offers the deepest and most comprehensive technical investigation of a variety of different neo-Fregean approaches.

Neo-Fregeanism began with Crispin Wright (*Frege's Conception of Numbers as Objects*, 1983) who suggested that the problem posed by Russell's paradox be evaded by making do with the first step of Frege's approach, abandoning altogether the second step and its inconsistent theory of extensions. This approach is made possible by two relatively recent technical discoveries. The first discovery is that (HP), unlike (V), is consistent. More precisely, let *Frege Arithmetic* be the second-order theory, with (HP) as its sole non-logical axiom. Frege Arithmetic can then be shown to be consistent if and only if second-order Peano Arithmetic is. The second discovery is that Frege Arithmetic and some very natural definitions suffice to derive all the axioms of second-order Peano Arithmetic. This result is known as *Frege's Theorem*. It is an amazing result. For more than a century now, informal arithmetic has almost without exception been given some Peano-Dedekind style axiomatization, where the natural numbers are regarded as finite ordinals, defined by their position in an omega-sequence. Frege's Theorem shows that an alternative and conceptually completely different axiomatization of arithmetic is possible, based on the idea that the natural numbers are finite cardinals, defined by the cardinalities of the concepts whose numbers they are.

Technically speaking, the neo-Fregean foundation of arithmetic is thus a success: it is consistent and strong...