e: The Master of All

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Laplace's dictum may rightfully be transcribed as: "Study e, study e. It is the master of all."

Just as Gauss earned the moniker Princeps mathematicorum amongst his contemporaries [1], e may be dubbed Princeps constantium symbolum. Certainly, Euler would concur, or why would he have endowed it with his own initial [2]? The historical roots of e have been exhaustively traced and are readily available [3, 4, 5]. Likewise, certain fundamental properties of e such as its limit definition, its series representation, its close association with the rectangular hyperbola, and its relation to compound interest, radioactive decay, and the trigonometric and hyperbolic functions are too well known to warrant treatment here [6, 7].

Rather, the focus of the present article is on matters related to e which are not so widely appreciated or at least have never been housed under one roof. When I do indulge in reviewing well-known facts concerning e, it is to forge a link to results of a more exotic variety.

Occurrences of e throughout pure and applied mathematics are considered; exhaustiveness is not the goal. Nay, the breadth and depth of our treatment of e has been chosen to convey the versatility of this remarkable number and to whet the appetite of the reader for further investigation.

The Cast

Unlike its elder sibling \(\pi\), e cannot be traced back through the mists of time to some prehistoric era [8]. Rather, e burst into existence in the early seventeenth century in the context of commercial transactions involving compound interest [5]. Unnamed usurers observed that the profit from interest increased with increasing frequency of compounding, but with diminishing returns.

Thus, e was first conceived as the limit

\[
e = \lim_{n \to \infty} \left(1 + 1/n\right)^n = 2.718281828459045 \cdots, \tag{1}
\]

although its baptism awaited Euler in the eighteenth century [2]. One might naively expect that all that could be gleaned from equation (1) would have been mined long ago. Yet, it was only very recently that the asymptotic development

\[
\hat{e}_n = (1 + 1/n)^n = \sum_{k=0}^{\infty} \frac{e_k}{n^k};
\]

\[
e = e \sum_{k=0}^{\infty} \frac{S_k(n+k,k)}{(n+k)!} \frac{(-1)^k}{k!}, \tag{2}
\]

with \(S_k\) denoting the Stirling numbers of the first kind [9], was discovered [10].

Although equation (1) is traditionally taken as the definition of e, it is much better approximated by the limit

\[
e = \lim_{n \to \infty} \left[\frac{(n + 2)^n}{(n + 1)^{n+1}} - \frac{(n + 1)^{n+1}}{n^n}\right] \tag{3}
\]

discovered by Brothers and Knox in 1998 [11]. Figure 1 displays the sequences involved in equations (1) and (3). The superior convergence of equation (3) is apparent.

In light of the fact that both equations (1) and (3) provide rational approximations to e, it is interesting to note that \(\frac{878}{325} = 2.71826 \cdots\) provides the best rational approximation to e, with a numerator and denominator of fewer than four decimal digits [12]. (Note the palindromes!) Considering that the fundamental constants of nature (speed of light in vacuo, mass of the electron, Planck's constant, and so on) are known reliably to only six decimal digits, this is remarkable accuracy indeed. Mystically, if we simply delete...
the last digit of both numerator and denominator then we obtain \( \frac{80}{57} \), the best rational approximation to \( e \) using fewer than three digits [13]. Is this to be regarded as a singular property of \( e \) or of base 10 numeration?

In 1669, Newton published the famous series representation for \( e \) [14],

\[
e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{1} + \frac{1}{2} \cdot \left(1 + \frac{1}{3} \cdot \left(1 + \frac{1}{4} \cdot \left(1 + \frac{1}{5} \cdot \left(1 + \cdots \right)\right)\right)\right)
\]  

(4)

established by application of the binomial expansion to equation (1). Many more rapidly convergent series representations have been devised by Brothers [14] such as

\[
e = \sum_{k=0}^{\infty} \frac{2k+2}{(2k)!}.
\]  

(5)

Figure 2 displays the partial sums of equations (4) and (5) and clearly reveals the enhanced rate of convergence. A variety of series-based approximations to \( e \) are offered in [15].

Euler discovered a number of representations of \( e \) by continued fractions. There is the simple continued fraction [16]

\[
e = 2 + \frac{1}{1 + \frac{2}{1 + \frac{1}{2 + \frac{1}{1 + \cdots}}}}.
\]  

(6)

or the more visually alluring [5]

\[
e = 2 + \frac{1}{1 + \frac{2}{1 + \frac{3}{1 + \frac{4}{1 + \cdots}}}}.
\]  

(7)

In 1655, John Wallis published the exhilarating infinite product

\[
\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{10}{11} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{14}{13} \cdot \frac{14}{15} \cdot \cdots
\]  

(8)

However, the world had to wait until 1980 for the “Pippenger product” [17]

\[
\frac{e}{2} = \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{10}{11} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{14}{13} \cdot \frac{14}{15} \right) \cdots
\]  

(9)