ON THE IMBEDDING PROBLEM WITH NONCOMMUTATIVE KERNEL OF ORDER $p'$. III

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In the present paper we continue to investigate conditions for solvability of the imbedding problem for number fields, whose kernel is a non-Abelian group of order $p'$. We retain the approach and the notation of the previous papers [1-4] as far as possible. As a kernel of the imbedding problem, we shall consider those groups of order $p'$ that are isomorphic to the direct product of a non-Abelian group of order $p^2$ and the group of order $p$ with a distinguished generator $c$. We observe that there exist two nonequivalent non-Abelian groups of order $p^2$, namely, the groups on the generators $a, b$ with relations $a^{p^2} = 1, b^{p^2} = 1, [a, b] = a^p$ in the first case, and $a^{p^2} = 1, b^{p^2} = 1, [a, b, a, b] = 1, [a, b, b] = 1$ in the second, where the square brackets denote the commutator of the correspondent elements.

We recall some notation: $K/k$ is a normal $p$-extension of algebraic number fields with Galois group $G$. Moreover, we are given a group extension $1 \rightarrow B \rightarrow G \rightarrow F \rightarrow 1$ with a non-Abelian kernel of order $p'$. Let $\phi$ denote the homomorphism of the group $G$ onto $F$. Thus, there arises the imbedding problem $(K/k, G, \phi)$ of constructing a field (Galois algebra) $L$ over the field $k$ with Galois group $G$ such that the restriction of every automorphism $\phi$ of $G$ to the subfield $K$ coincides with the image $\phi(g)$.

At every point $v$ of the field $k$ a local problem $(K_v/k_v, \phi_v)$ with $K_v \cong k_v^{p^2}$ corresponds to the given imbedding problem $(K/k, G, \phi)$. The associated problem $(K/k, G/B', \phi')$, which is obtained from the given one by factoring the kernel by the commutant subgroup, is also defined (here $\phi'$ is the natural homomorphism of the group $G/B'$ onto $F'$).

**THEOREM.** The imbedding problem $(K/k, G, \phi)$ is solvable if and only if the associated problem $(K/k, G/B', \phi')$ is solvable for every point $v$ of the field $k$, and the Abelian problem $(K/k, G/B', \phi')$ is solvable.

**Proof.** The necessity of the condition is obvious. Conversely, suppose that all associated local problems and the Abelian problem are solvable. We observe that one can assume that the field $k$ contains a primitive root $\zeta$ of 1 of degree $p$, since otherwise the assertion of the theorem could be deduced immediately from a theorem of Neukirch (cf. [7]).


LITERATURE CITED

Direct computation shows that the Sylow $p$-subgroup $\text{Aut}_p$ of the automorphism group of the group $\mathfrak{C}$ has three generators $\sigma, \tau, \omega$, which act on elements of $\mathfrak{C}$ according to the rules:

- $a^{\sigma} = a b$
- $b^{\sigma} = b$  
- $c^{\sigma} = c$

- $a^{\tau} = a$
- $b^{\tau} = a c$
- $c^{\tau} = c$

- $a^{\omega} = a$
- $b^{\omega} = b$
- $c^{\omega} = c$

and the relations $\sigma^2 = 1$, $\tau^p = 1$, $\omega^p = 1$, $\sigma^2 = 1$, $\tau^2 = 1$, $\tau \sigma = \sigma \tau$.

The Sylow $p$-subgroup $\text{Aut}_p$ of the group of external automorphisms of $\mathfrak{C}$ is generated by the automorphisms $\sigma^2, \tau, \omega^2$, which are the images of the automorphisms $\sigma, \tau, \omega$ respectively, and satisfy the additional relation $\sigma^2 \tau^2 = 1$. There exists a natural homomorphism $\varphi$ of the group $\mathfrak{C}$ into $\text{Aut}_p$. Let $\eta_1$ be the factor group of the group $\mathfrak{C}$ by the commutant $\mathfrak{C}^'$, $\eta_1 = \mathfrak{C} / \mathfrak{C}^'$, and let $\varphi_1$ be the natural map of the group $\mathfrak{C}$ onto $\eta_1$.

**Lemma.** The associated embedding problem $(K / k, \eta_1, \varphi_1, \omega)$ is solvable except, perhaps, when $p^2 \equiv 2$ and the image $\varphi(\mathfrak{C})$ is generated by the elements $\sigma \tau$, $\sigma \tau \omega$, with $\omega \in \mathfrak{C}^p$.

**Proof.** We observe that the local problem $(K / l, \mathfrak{C} / \mathfrak{C}^p, \varphi_1, \omega)$, which is also associated with the embedding problem $(K / \mathfrak{C} / k, \mathfrak{C}, \varphi, \omega)$, so, by the hypothesis, for every point $P$ the embedding problem $(K / \mathfrak{C} / k, \mathfrak{C}, \varphi, \omega)$ is solvable.

Further, we use a theorem of Yakovlev (cf. [6]). Let $\mathfrak{H}$ denote the factor group of the group $\mathfrak{C}$ by the subgroup of elements acting trivially on the module $\text{Hom}_{\mathfrak{C}}(\mathfrak{C}, k)$. Then it suffices to prove injectivity of the natural map $\mathfrak{H} / \text{Hom}_{\mathfrak{C}}(\mathfrak{C}, k)$ is generated by the images of $\mathfrak{C} / \mathfrak{C}^p$, respectively, in the group $\mathfrak{C}$, and suppose that the automorphisms $\alpha, \beta, \gamma$ are given by the equations:

- $x^\alpha = x$  
- $x^\beta = x$  
- $x^\gamma = x$

We can see immediately that the automorphism group $\mathfrak{H}$ of the group $\mathfrak{C}$ is generated by the automorphisms $\sigma^2$ and $\tau$ acting according to the formulas:

- $x^{\sigma} = x_1$  
- $x^{\tau} = x_2$  
- $x^{\sigma \tau} = x_3$  
- $x^{\tau \sigma} = x_4$  
- $x^{\sigma^2} = x_5$  
- $x^{\tau^2} = x_6$  
- $x^{\omega} = x_7$  
- $x^{\omega^2} = x_8$

The group $\mathfrak{H}$ is a subgroup of the group generated by the elements $\sigma$ and $\tau$. We consider the case when $\mathfrak{H}$ coincides with this group. Let $h$ be a locally splitting cocycle in $\text{C}(\mathfrak{H})$, then, considering the restrictions of $h$ on the subgroups $\mathfrak{C}$ and $\mathfrak{C}^p$, we have $h(\sigma) = x_1$, $h(\tau) = x_2$, i.e., $h$ is a coboundary. Any other case in which $\mathfrak{H}$ is a subgroup of the group $\mathfrak{H}$ can be considered similarly, except the case in which $\mathfrak{H}$ is generated by the elements $\sigma \tau$, $\sigma \tau \omega$, with $\omega \in \mathfrak{C}^p$. In this case one can see that the map $\varphi$ is not injective. The lemma is proved.

We can pass from the given problem to an equivalent one, for which the associated problem obtained by factoring the kernel by the commutant is semidirect (cf. [4], Sec. 3°). In the sequel, we shall use the original notation for this new problem. Therefore, we can assume that the extension $\mathfrak{C} / \mathfrak{C}^p$ is semidirect, and we can consider the action of the