ON ISOMORPHISMS OF CONTINUOUS FUNCTION SPACES

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ABSTRACT

The following theorem, which strengthens the classical theorem of Stone, is proved: If there is an isomorphism \( \varphi \) of \( C(X) \) onto \( C(Y) \) (\( X, Y \) - compact) with \( \| \varphi \| \cdot \| \varphi^{-1} \| < 2 \), then \( X \) and \( Y \) are homeomorphic.

1. Introduction and statement of the theorem. Let \( X, Y \) be compact Hausdorff spaces, and \( C(X), C(Y) \) — the corresponding Banach spaces of continuous real-valued functions on \( X \) and \( Y \), with the maximum norm. Suppose there is a linear isomorphism \( \phi \) of \( C(X) \) onto \( C(Y) \). If \( \phi \) is an isometry, then by a theorem of Stone [2] \( X \) and \( Y \) are homeomorphic. If we only require that \( \phi \) is a topological linear isomorphism, then \( X \) and \( Y \) may fail to be homeomorphic. Let us give some simple examples:

EXAMPLE A. \( X \) is a sequence of isolated points: \( x_1, x_2, \ldots \) with two limit points: \( x_{2m} \to x^0, x_{2m+1} \to x^1 \); \( y \) is a sequence of isolated points: \( y_0, y_1, y_2, \ldots \) with one limit point: \( y_n \to y \).

For \( f \in C(X) \) define \( \phi f \):
\[
(\phi f)(y_0) = f(x^0) - f(x^1), \quad (\phi f)(y_{2m}) = f(x_{2m}) - \frac{1}{2}[f(x^0) - f(x^1)], \\
(\phi f)(y_{2m+1}) = \frac{1}{2}[f(x^0) - f(x^1)] + f(x_{2m+1}), \\
((\phi f)(y) = \frac{1}{2}[f(x^0) + f(x^1)]
\]

\( \phi \) is a linear isomorphism of \( C(X) \) onto \( C(Y) \), with \( \| \phi \| = 2, \| \phi^{-1} \| = \frac{1}{2} \).

EXAMPLE B. \( X \) is the unit interval \([0,1]\); \( Y \) is the unit circle \( \{e^{2\pi ix} : 0 \leq x < 1\} \) plus the isolated point 0. For \( f \in C(X) \) define \( \phi f \):
\[
(\phi f)(0) = f(1) - f(0), \quad (\phi f)(e^{2\pi ix}) = f(x) + (\frac{1}{2} - x)[f(1) - f(0)].
\]

\( \phi \) is an isomorphism of \( C(X) \) onto \( C(Y) \); here too \( \| \phi \| = 2, \| \phi^{-1} \| = \frac{1}{2} \).

EXAMPLE C. \( X = A \cup B \) where \( A, B \) are compact and \( A \cap B = \{p\} \); \( Y = A \cup B' \) where \( B' \) is homeomorphic to \( B \) and \( A \cap B' = \{q\} \).

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For \( f \in C(X) \) define:

\[
(\phi f)(y) = \begin{cases} 
  f(x) - \frac{1}{2} [f(q) - f(p)] & \text{if } y = x \in A \\
  f(x) + \frac{1}{2} [f(q) - f(p)] & \text{if } y = x' \in B
\end{cases}
\]

\((x \to x')\) is the homeomorphism from \( B \) to \( B' \).

In this example \( \|\phi\| = \|\phi^{-1}\| = 2 \).

\[
\begin{array}{c}
A \\
q \quad p
\end{array}
\quad X
\quad Y
\begin{array}{c}
B \\
q \quad p
\end{array}
\]

or, with some more imagination:

\[
\begin{array}{c}
X \\
\circ
\quad q
\quad p
\end{array}
\quad Y
\begin{array}{c}
\circ \\
\quad q
\quad p
\end{array}
\]

I do not know any example of nonhomeomorphic \( X, Y \) such that \( \|\phi\| \|\phi^{-1}\| < 3 \). However we were able to prove only the following generalization of Stone's theorem:

**Theorem.** If \( X, Y \) are compact Hausdorff spaces, and \( \phi \) a linear isomorphism of \( C(X) \) onto \( C(Y) \) with \( \|\phi\| \|\phi^{-1}\| < 2 \), then \( X \) and \( Y \) are homeomorphic.

Thus leaving open the problem: Do there exist nonhomeomorphic compact Hausdorff \( X, Y \) and an isomorphism \( \phi \) of \( C(X) \) onto \( C(Y) \) with \( 2 \leq \|\phi\| \|\phi^{-1}\| < 3 \)?

**2. Proof of the theorem.** Let \( \phi \) be an isomorphism of \( C(X) \) onto \( C(Y) \). Denote \( \|\phi\| = \alpha, \|\phi^{-1}\| = \beta \), and assume \( \alpha \beta < 2 \). Denote also \( C_+(X) = \{f \in C(X); f \geq 0, \|f\| = 1\} \) and similarly \( C_+(Y) \). 1 is the constant function 1.

**Lemma 1.** There is an isomorphism \( \hat{\phi} \) of \( C(X) \) onto \( C(Y) \) satisfying:

(a) \( \|\hat{\phi}\| = \alpha \); (b) \( \|\hat{\phi}^{-1}\| = \beta \); (c) \( (\hat{\phi}1)(y) \geq \alpha(2 - \alpha \beta) \) for each \( y \in Y \);

(d) \( (\hat{\phi}^{-1}1)(x) \geq \beta(2 - \alpha \beta) \) for each \( x \in X \);

(e) for each \( f \in C_+(X) \) there is some \( y \in Y \) such that \( (\phi f)(y) \geq \frac{1}{\beta} \);

(f) for each \( g \in C_+(Y) \) there is some \( x \in X \) such that \( (\phi^{-1}g)(x) \geq \frac{1}{\alpha} \).

**Proof.** Let \( y_0 \in Y \), and \( t = (\phi 1)(y_0) \). If \( |t| < \alpha \), take any \( 0 < \varepsilon < \alpha - |t| \) and a neighborhood \( N \) of \( y_0 \) in which \( |(\phi 1)(y) - t| < \varepsilon \). Let \( g \in C_+(Y) \) be such that \( g(Y \setminus N) = 0 \). Consider the functions: \( G_1(y) = (\phi 1)(y) + (\alpha - t - \varepsilon)g(y), \)

\( G_2(y) = (\phi 1)(y) - (\alpha + t - \varepsilon))g(y) \).