BOUNDARY SMOOTHNESS PROPERTIES OF Lipα ANALYTIC FUNCTIONS

By
D. J. LORD* AND A. G. O’FARRELL†

Abstract. Let $U$ be an open set and $b \in \partial(U)$. Let $0 < \alpha < 1$. Let $A(U)$ denote the space of Lipα functions that are analytic on $U$, and $a(U)$ the subspace $\text{lip}_\alpha \cap A(U)$. The space $a(U \cup \{b\})$, consisting of the functions that are analytic near $b$, is dense in $a(U)$. Let $k$ be a natural number. We say that $a(U)$ admits a $k$-th order continuous point derivation (cpd) at $b$ if the functional $f \mapsto f^{(k)}(b)$ is continuous on $a(U \cup \{b\})$, with respect to the Lipα norm.

Theorem $a(U)$ admits a $k$-th order cpd at $b$ if and only if

$$\sum_{n=1}^{\infty} 2^{(k+1)n} M_\beta^{1+\alpha}(A_n(b) \sim U) < +\infty.$$ 

Here $M_\beta$ denotes $\beta$-dimensional lower Hausdorff content, and $A_n(b)$ denotes the annulus

$$\{z \in \mathbb{C} : |z-b| \in [2^{-n-1}, 2^{-n}]\}.$$ 

There is a weak-star topology on $A(U)$, and the space $A(U \cup \{b\})$ is weak-star dense in $A(U)$. We say that $A(U)$ admits a $k$-th order weak-star cpd at $b$ if the functional $f \mapsto f^{(k)}(b)$ is weak-star continuous on $A(U \cup \{b\})$.

Theorem $A(U)$ admits a $k$-th order weak-star cpd at $b$ if and only if

$$\sum_{n=1}^{\infty} 2^{(k+1)n} M_\beta^{1+\alpha}(A_n(b) \sim U) < +\infty.$$ 

This time, $M_\beta$ denotes ordinary $\beta$-dimensional Hausdorff content.

1. Introduction

Let $0 < \alpha < 1$. For $E \subset \mathbb{C}$ and $f : E \to \mathbb{C}$ let

$$\|f\|_{\text{Lip}_\alpha(E)} = \sup \left\{ \frac{|f(z) - f(w)|}{|z - w|^{\alpha}} : z \neq w \right\}.$$ 

We call $\|f\|_{\text{Lip}_\alpha(E)}$ the Lipα($E$) seminorm of $f$. We denote

$$\text{Lip}_\alpha(E) = \{ f \in \mathbb{C}^E : \|f\|_{\text{Lip}_\alpha} < +\infty \}.$$ 

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This is a Banach space when endowed with the norm
\[
\|f\|_{\text{Lip}_\alpha} = |f(b)| + \|f\|_{\text{Lip}_\alpha},
\]
where \(b\) is any fixed point of \(E\). We abbreviate \(\text{Lip}_\alpha(\mathbb{C})\) to \(\text{Lip}_\alpha\). The subspace \(\text{lip}_\alpha \subset \text{Lip}_\alpha\) consists of those \(f \in \text{Lip}_\alpha\) such that
\[
\lim_{\delta \downarrow 0} \sup_{0 < |z-w| < \delta} \frac{|f(z) - f(w)|}{|z-w|^{\alpha}} = 0.
\]

For open sets \(U \subset \mathbb{C}\) we denote
\[
A(U) = \{f \in \text{Lip}_\alpha : \partial f = 0 \text{ on } U\},
\]
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\]
Here \(\partial f\) denotes the distributional \(\partial\)-derivative
\[
\partial f = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).
\]

In view of Weyl’s Lemma, “\(\partial f = 0\) on \(U\)” is a way of saying that the restriction \(f|U\) is an analytic function.

This paper is about the extent to which the functions belonging to \(A(U)\) or \(a(U)\) may be better-behaved at points of \(\text{bdy}U\) than are typical elements of \(\text{Lip}_\alpha\) or \(\text{lip}_\alpha\). Specifically, we consider the question of the existence of bounded point derivations. We will explain this concept shortly. First, we review some classical facts.

Suppose \(b\) is an isolated point of \(\text{bdy}U\). Then, since the elements \(f \in A(U)\) are bounded and analytic on a deleted neighbourhood of \(b\), it follows that they extend analytically across \(b\), and since they are continuous, they are already analytic on \(U \cup \{b\}\).

Similarly, if a line segment \(I\) forms a relatively-open subset of \(\mathbb{C} \sim U\), then each function \(f \in A(U)\) extends analytically across \(I\).

These facts may be rephrased in terms of the concept of \(\partial\)-\(\text{Lip}_\alpha\)-null set:

A compact \(K \subset \mathbb{C}\) is said to be \(\partial\)-\(\text{Lip}_\alpha\)-null if
\[
A(U \sim K) = A(U)
\]
whenever \(U \subset \mathbb{C}\) is open.

Singletons and line segments are \(\partial\)-\(\text{Lip}_\alpha\)-null.