RUELLE OPERATORS WITH RATIONAL WEIGHTS FOR JULIA SETS

By

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Abstract. Let $R$ be a rational function with nonempty set of normality that consists of basins of attraction only and let

$$(L_Qg)(z) = \sum_{R(w) = z} Q(w)g(w)$$

be a Ruelle operator with a rational weight $Q$ which acts in a space of locally analytic functions on the Julia set of $R$. We obtain explicit expressions for equations for the eigenvalues and study the structure of eigenfunctions of $L_Q$ and its adjoint operator. Some applications to operators important for measurable dynamics of rational functions will be considered.

Introduction and notation

Ruelle operators (or transfer operators) with positive weights are one of the main tools in the thermodynamic formalism that was developed in the works of Sinai, Ruelle and Bowen (e.g. see Ruelle [9],[10]). In a number of recent works, spectral properties of Ruelle operators with more general weights were investigated (see Parry and Pollicott [7], Mayer [5],[6], Ruelle [11] and references there). As was noted by Sullivan, the famous linear Feigenbaum operator may be reduced to a Ruelle operator with complex analytic weight which has alternating signs on the real axis (see Jiang, Morita, Sullivan [3], Pollicott [8]). Using methods of perturbation theory, we obtained in [4] explicit expression for the Fredholm determinant of Ruelle operators for an iterated rational function $R(z)$ with a real Julia set and for the particular weight $(R')^{-2}$. In the present paper, we use a different approach to study Ruelle operators with rational weights for an arbitrary iterated rational function with nonempty set of normality that consists of basins of attraction only.

Let $R$ be a rational function of degree $d \geq 2$ with set of normality (Fatou set) $F$, and let $J = \mathbb{C} \setminus F$ be the Julia set of $R$. We assume that $R$ satisfies the following conditions:

(a) $F \neq \emptyset$;
(b) every point of $F$ lies in a basin of attraction of some attracting or superattracting periodic orbit of $R$.

In particular, the expanding rational functions satisfy these conditions.
It follows immediately from Sullivan's classification of components of the Fatou set (see, for example, Beardon [1]) that the condition (b) may be replaced by

(b1) every critical point of \( R \) lies either in a basin of attraction of an attracting or superattracting periodic orbit of \( R \) or on the Julia set \( J \).

Let \( Q \) be a rational function without poles on \( J \). We consider the Ruelle operator

\[
(L_Q f)(z) = \sum_{R(w) = z} Q(w) f(w)
\]

in the space of locally analytic functions on \( J \). In this space \( L_Q \) has only point spectrum with the only possible point of condensation at the origin; the adjoint operator \( L_Q^* \) has the same spectrum (see Section 1). The main goal of the present paper is to show that spectral characteristics of this operator (equations for the eigenvalues and the structure of eigenfunctions) may be evaluated rather explicitly.

In Theorem 1 we find possible singularities of eigenfunctions of the operator \( L_Q \). In particular, our considerations show that if \( R \) has only superattracting periodic points, then \( L_Q \) has a finite spectrum and all its eigenfunctions (corresponding to nonzero eigenvalues) are rational functions.

One of the ideas of our approach is to pass to the adjoint operator \( L_Q^* \). In Theorem 2 we prove that a function \( f \) holomorphic outside \( J \) is an eigenfunction of \( L_Q^* \) if and only if it satisfies a functional equation

\[
f(z) - \frac{1}{\rho} f(Rz) R'(z) Q(z) = \phi(z),
\]

where \( \phi \) is a rational function without poles on \( J \). In Section 3 we propose a general scheme of solution of Eq. (0.2).

An essential part of the paper consists of examples in which all calculations may be completed. In Section 2 we examine the case of a rational function \( R \) that preserves the upper and lower halfplanes and has an attracting fixed-point at infinity, \( Q = 1/R' \).

In Section 4 we consider the case of polynomials \( R \) and \( Q \), and in Section 5 we analyse the case of a quadratic expanding polynomial \( R \) and \( Q = 1/(R')^2 \). In the latter case we also study a problem of localization of the spectrum.

In the last Section 6 we investigate the case of the nonrational weight \( Q = |R'|^{-s}, 0 < s < 2 \), for a quadratic polynomial \( R \) with real Julia set.

Now we introduce some notation. We denote by \( \mathfrak{A} \) the set of attracting and superattracting periodic points of \( R \). By \( R(G) \) and \( R^{-1}(G) \) we denote the image and the full preimage of the set \( G \) under \( R \) and by \( R^n \) we denote the \( n \)-th iterate of \( R \).