PSEUDO-CHEBYSHEV SUBSPACES IN $L^1$

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Abstract. We give various characterizations of pseudo-Chebyshev subspaces in the spaces $L^1(S, \mu)$ and $C(T)$.

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1. Introduction and Preliminaries

Let $(S, \mu)$ be a positive measure space and let $L^1(S, \mu)$ be the Banach space of all complex-valued measurable functions (the equivalence classes) defined on $S$ and equipped with the norm

$$
\|f\|_1 = \int_S |f| d\mu, \quad f \in L^1(S, \mu).
$$

We denote by $L^\infty(S, \mu)$ the Banach space of all essentially bounded complex-valued measurable functions defined on $S$ and equipped with the norm

$$
\|f\|_\infty = \text{ess sup}\{|f(s)|; \ s \in S\}, \quad f \in L^\infty(S, \mu).
$$

A measure space $(S, \mu)$ is called a $\sigma$-finite measure space if there exists a sequence $\{A_n\}_{n \geq 1}$ of measurable subsets of $S$ such that $S = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < +\infty$ for all $n = 1, 2, \ldots$. Let $T$ be a compact Hausdorff topological space and let $C(T)$ be the Banach space of all complex-valued continuous functions defined on $T$ and equipped with the supremum norm ($\|f\| = \sup_{t \in T} |f(t)|, \quad f \in C(T)$).

Let $X$ be a (complex or real) normed linear space and let $Y$ be a linear subspace of $X$. A point $y_0 \in Y$ is said to be a best approximation for $x \in X$
if
\[ \|x - y_0\| = d(x, Y) = \inf \{\|x - y\| : y \in Y\}. \]
If each \( x \in X \) has at least one best approximation in \( Y \), then \( Y \) is called a proximinal subspace of \( X \). If each \( x \in X \) has a unique best approximation in \( Y \), then \( Y \) is called a Chebyshev subspace of \( X \).

For \( x \in X \), put
\[ P_Y(x) = \{y \in Y : \|x - y\| = d(x, Y)\}. \]

It is clear that \( P_Y(x) \) is a closed bounded convex subset of \( X \) for every \( x \in X \).

A linear subspace \( Y \) of a normed linear space \( X \) is called pseudo-Chebyshev if \( P_Y(x) \) is a non-empty and finite-dimensional set for every \( x \in X \) (see [5]).

For an arbitrary non-empty convex set \( A \) in a normed linear space \( X \), we denote by \( \ell(A) = \{\alpha x + (1 - \alpha) y : x, y \in A; \ \alpha \ \text{is scalar}\} \) the linear manifold spanned by \( A \). For every fixed \( y \in A \) the set \( \ell(A) - y = \{x - y : x \in \ell(A)\} \) is a linear subspace of \( X \) and satisfying \( \ell(A) - y = \ell(A - y) \).

The dimension of an arbitrary convex set \( A \) in \( X \) is defined by
\[ \dim A = \begin{cases} \dim \ell(A), & A \neq \emptyset \\ -1, & A = \emptyset. \end{cases} \]

Then, for every \( y \in A \) we have
\[ \dim A = \dim \ell(A) = \dim[\ell(A) - y] = \dim \ell(A - y) = \dim(A - y). \]

(For more details see [6].)

For an easy reference, we gather some known results which will need in the proof of the main results.

**Lemma 1.1 ([4, 6; Theorem 1.1]).** Let \( Y \) be a linear subspace of a normed linear space \( X \) and let \( x \in X \setminus \overline{Y} \), \( y_0 \in Y \). Then \( y_0 \in P_Y(x) \) if and only if there exists \( f \in X^* \) such that \( \|f\| = 1 \), \( f|Y = 0 \) and \( f(x - y_0) = \|x - y_0\| \).

**Lemma 1.2 ([1; Theorem 2.1]).** Let \( W \) be a proximinal linear subspace of a normed linear space \( X \). Then the following are equivalent:

a) \( W \) is pseudo-Chebyshev in \( X \).

b) There do not exist \( f \in X^* \), \( x_0 \in X \) and infinitely many linearly independent elements \( x_1, x_2, \ldots \) in \( X \) with \( x_0 - x_n \in W \) \((n = 1, 2, \ldots)\) such that \( \|f\| = 1 \), \( f|W = 0 \) and \( f(x_n) = \|x_n\| \) for all \( n = 0, 1, 2, \ldots \).

c) There do not exist \( f \in X^* \), \( x_0 \in X \) and infinitely many linearly independent elements \( g_1, g_2, \ldots \) in \( W \) such that \( \|f\| = 1 \), \( f|W = 0 \) and \( f(x_0) = \|x_0\| = \|x_0 - g_n\| \) for all \( n = 1, 2, \ldots \).