A PROPERTY OF MEAN OF HARMONIC FUNCTIONS.

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The following theorem was proved by Leifur Ásgeirsson 1): In the space of n dimensions be given two confocal (n - 1)-dimensional ellipsoids $\varepsilon_1$ and $\varepsilon_2$. Their interior be denoted by $\mathcal{I}_1$ and $\mathcal{I}_2$. Be further $u(x_1, x_2, \ldots, x_n)$ a harmonic function in a domain of the $(x_1, x_2, \ldots, x_n)$ space which contains $\mathcal{I}_1$ and $\mathcal{I}_2$. Then the arithmetical means of $u(x_1, x_2, \ldots, x_n)$ in the domains $\mathcal{I}_1$, $\mathcal{I}_2$ respectively are equal.

This statement is a secondary result of a theorem proved in the same paper. Now the validity of the theorem quoted above is to be shown more directly. For sake of simplicity instead of $n$ coordinates we will use only three, namely $x, y, z$.

Consider at first the functions

$$U(x, y, z) = G^m_n(x, y, z) \quad (n = 0, 1, 2, \ldots; m = 1, 2, \ldots, 2n + 1)$$

where $G^m_n(x, y, z)$ is one of the $2n + 1$ polynomials of degree $n$ in $x, y, z$ which are harmonic functions belonging to the set of confocal quadrics defined by the equation

$$(1) \quad \frac{x^2}{a^2 + \theta} + \frac{y^2}{b^2 + \theta} + \frac{z^2}{c^2 + \theta} = 1 \quad (a > b > c).$$

We shall show that the arithmetical mean of any ellipsoidal harmonic (excepted three one's) taken in any ellipsoid of the set (1) is equal to 0.

This statement is obvious for any ellipsoidal harmonic of the 2nd, 3rd and 4th species, for these functions are odd functions of at least one of the variables $x, y, z$. Further if $V(A)$ means the interior of the ellipsoid with parameter $A$, the region

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\( P(\Lambda) \) is symmetric with respect to each of the planes \( x = 0, y = 0, z = 0 \). So
\[
\int \int \int_{P(\Lambda)} f(x, y, z) \, dV = \int \int \int_{P(\Lambda)} f(\pm x, \pm y, \pm z) \, dV
\]
the signs \( \pm \) being chosen ad libitum. Out of this follows
\[
\int \int \int_{P(\Lambda)} f(x, y, z) \, dV = \int \int \int_{P(\Lambda)} \frac{1}{2} [f(x, y, z) + f(\pm x, \pm y, \pm z)] \, dV.
\]
Now if \( f(x, y, z) \) is an odd function of any of the three variables, the double signs can be chosen in the manner that the integrand of the right side vanishes. Hence the value of the whole integral is 0.

Now turn to the confocal harmonics of the first species. Introducing confocal coordinates \( \lambda, \mu, \nu \) defined by the roots of equation (1) for \( \theta \) we get
\[
G_n^m(x, y, z) = E_n^m(\lambda) E_n^m(\mu) E_n^m(\nu),
\]
where \( E_n^m(\theta) \) is Lamé's polynomial. Further we make the usual agreement \( \lambda > \mu > \nu \). Using these coordinates for calculating the value of the integral \( I_n^m(\Lambda) \) of \( G_n^m(x, y, z) \) in the volume of an ellipse, we get
\[
I_n^m(\Lambda) = \int \int \int_{P(\Lambda)} G_n^m \, dV
\]
\[
= \int_{-\lambda}^{\lambda} \int_{-\mu}^{\mu} \int_{-\nu}^{\nu} E_n^m(\lambda) E_n^m(\mu) E_n^m(\nu) \left[ \frac{(\lambda - \mu)(\mu - \nu)(\nu - \lambda)}{\Delta(\lambda) \Delta(\mu) \Delta(\nu)} \right] \, d\lambda \, d\mu \, d\nu
\]
\[
= \int \int \int_{-\lambda}^{\lambda} E_n^m(\lambda) \left[ \int_{-\mu}^{\mu} \int_{-\nu}^{\nu} E_n^m(\mu) E_n^m(\nu) \left( \frac{\mu - \nu}{\Delta(\mu) \Delta(\nu)} \right) \, d\mu \, d\nu \right] \, d\lambda.
\]
where \( \Delta(\theta) \) is an abbreviation for \( \sqrt{(a^2 + \theta)(b^2 + \theta)(c^2 + \theta)} \).

We are going to show that this integral vanishes excepted but a few cases. Our starting point will be the so-called « relation of orthogonality » of the Lamé's functions \( ^{2) \)}
\[
\int_{-\lambda}^{\lambda} \int_{-\mu}^{\mu} E_n^m(\lambda) E_n^m(\mu) \left( \frac{\mu - \nu}{\Delta(\mu) \Delta(\nu)} \right) \, d\mu \, d\nu = 0,
\]
excepted when \( m = M, n = N \).

\( ^{2) \) See e. g. E. W. Hobson, The theory of spherical and ellipsoidal harmonics, Cambridge, 1931, p. 467.