The Dirichlet Problem

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Dirichlet’s problem is one of the fundamental boundary problems of physics. It appears in electrostatics, heat conduction, and elasticity theory and it can be solved in many ways. For the mathematicians of the nineteenth century it was a fruitful challenge that they met with new methods and sharper tools. I am going to give a sketch of the problem and its history.

Dirichlet did most of his work in number theory and is best known for having proved that every arithmetic progression \( a, a + b, a + 2b, \ldots \) contains infinitely many primes when \( a \) and \( b \) are relatively prime. In 1855, towards the end of his life, he moved from Berlin to Göttingen to become the successor of Gauss. In Berlin he had lectured on many things including the grand subjects of contemporary physics, electricity, and heat conduction. Through one of his listeners, Bernhard Riemann, Dirichlet’s name became attached to a fundamental physical problem. In bare mathematical terms it can be stated as follows.\(^2\)

A real-valued function \( u(x) = u(x_1, \ldots, x_n) \) from an open part \( \Omega \) of \( \mathbb{R}^n \) is said to be harmonic there if \( Au = 0 \) where \( \Delta = \partial^2_1 + \ldots + \partial^2_n, \partial_k = \partial/\partial x_k, \) is the Laplace operator. Dirichlet’s problem: given \( \Omega \) and a continuous function \( f \) on the boundary \( \Gamma \) of \( \Omega \), find \( u \) harmonic in \( \Omega \) and continuous in \( \Omega \cup \Gamma \) such that \( u = f \) on \( \Gamma \). When \( n = 1 \), the harmonic functions are of the form \( ax_1 + b \), and conversely, so that the reader may solve Dirichlet’s problem by himself when \( \Omega \) is an interval on the real axis. But if \( n > 1 \) we are in deep water. The physical examples that follow indicate that the problem is correctly posed in the sense that the solution is likely to exist and be uniquely determined by \( f \) and \( \Omega \), at least under some very light restrictions.

Gravitation and electrostatics

A gravitational or electric potential in \( \mathbb{R}^3 \),

\[
  u(x) = \int |x - y|^{-1} \rho(y)dy
\]

as he proved in 1810. Hence, outside the masses or charges, \( u \) is harmonic. This can also be seen by differentiating under the integral sign and noting that \( \Delta |x - y|^{-1} = 0 \) for fixed \( y \neq x \). Dirichlet’s problem here becomes: find a potential in a region outside the masses (or charges in the electric case) when its value is known on the boundary of the region.

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1 Expanded version of a lecture to a student audience at Lund University, Sweden.
2 The lack of precision at this point about the smoothness of functions and boundaries is intentional. Such vagueness was the rule in early nineteenth century mathematics (and in textbooks of not so long ago). Following the historical development, precision increases towards the end of the article.
Heat conduction

In his book *Theorie analytique de la chaleur* (1822), Fourier devised a mathematical model for the propagation of heat in a heat conducting body \( \Omega \). In this model, the temperature is represented by a function \( u(t, x) \) of time \( t \) and position \( x \) in \( \Omega \) which satisfies the heat equation \( \partial_t u = \Delta u \). In a state of equilibrium, \( u \) is independent of \( t \) and hence harmonic. Dirichlet's problem becomes: compute the equilibrium temperature in \( \Omega \) when its boundary \( \Gamma \) has a given time-independent temperature.

Elastic equilibrium

Let \( u(x_1, x_2) \) be a smooth function defined in an open bounded part \( \Omega \) of \( \mathbb{R}^2 \) and think of the graph of \( u \) as a thin elastic membrane in space. Keep \( u = f \) fixed over the boundary \( \Gamma \) of \( \Omega \). The potential energy of the membrane is supposed to be proportional to the stretching

\[
\int_\Omega \left( 1 + |\nabla u(x)|^2 \right)^{1/2} dx_1 dx_2,
\]

i.e., the area enlargement from \( u = \text{constant} \). For small \( \nabla u \), this is approximately half of the so-called Dirichlet integral

\[
\int_\Omega |\nabla u|^2 dx.
\]

If \( v \), another smooth function, vanishes on \( \Gamma \), an integration by parts gives

\[
\int_\Omega \left( \partial_1 u \partial_1 v + \partial_2 u \partial_2 v \right) dx = - \int_\Omega v \Delta u dx,
\]

so that

\[
\int_\Omega |\nabla (u + v)|^2 dx = \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx - 2 \int_\Omega v \Delta u dx.
\]

If \( u \) is harmonic, the last integral vanishes and we make the following observation, called Dirichlet's principle: of all functions \( w (= u + v) \) on \( \Omega \), equal to \( f \) on the boundary \( \Gamma \), the solution \( u \) of Dirichlet's problem has the least energy \( \int_\Omega |\nabla w|^2 dx \). By the laws of mechanics, this means that the corresponding membrane is in a state of equilibrium.

Poisson and Green

Before 1825, Poisson had found simple explicit solutions of Dirichlet's problem for a ball and a disk. For the ball \( |x| < R, n = 3 \),

\[
u(x) = (4\pi R)^{-1} \int_{|y|=R} \frac{(R^2 - |x|^2)|x - y|^{-3} f(y) dS(y)}{|y|},
\]

where \( dS(y) \) is the element of area on the sphere \( |y| = R \).

For the disk \( |x| < R, n = 2 \),

\[
u(x) = (2\pi R)^{-1} \int_{|y|=R} \frac{(R^2 - |x|^2)|x - y|^{-2} f(y) ds(y)}{|y|},
\]

where \( ds(y) \) is the element of arc length on the circle \( |y| = R \).

I will not go into Poisson's now obsolete proofs. Instead, I shall sketch how Green found an analogue of (2) for general regions. His construction is to be found in his famous 1828 paper entitled *An essay on the application of mathematical analysis to the theory of electricity and magnetism*, where he also proves the well-known Green's formula.

Green studied electrical potentials in three dimensions,

\[
V(x) = \int |x - y|^{-1} \rho(y) dy.
\]

Integrating the identity with arbitrary \( u \),

\[
|x - y|^{-1} \Delta u(y) = \text{div}_y (|x - y|^{-1} \nabla u(y))
\]

\[
- u(y) \nabla_y |x - y|^{-1},
\]

with respect to \( y \) over a bounded region \( \Omega \) minus a small ball around \( x \in \Omega \) and letting the radius of the ball tend to zero, he showed that

\[
4\pi u(x) = - \int |x - y|^{-1} \Delta u(y) dy + \int u(y) \frac{d}{dN} |x - y|^{-1} d\Gamma(y)
\]

\[
- \int |x - y|^{-1} \frac{d}{dN} u(y) dy d\Gamma(y),
\]

where \( \Gamma \) is the boundary of \( \Omega \), \( d\Gamma(y) \) its element of area, and \( d/dN \) the interior normal derivative at \( y \) (when \( x \) is outside of \( \Omega \), the right side is zero). If \( u \) is harmonic, \( \Delta u = 0 \), this formula gives us \( u \) in \( \Omega \) when we know \( u \) and \( du/dN \) on the boundary. Green observed that if one could find a function \( V(x, y) \), defined for \( x, y \) in \( \Omega \) and without singularities there, such that the functions \( y \to V(x, y) \) are harmonic and the function for fixed \( x \),

\[
G(x, y) = |x - y|^{-1} - V(x, y),
\]

(now called Green's function for \( \Omega \) with pole at \( x \in \Omega \)) vanishes on \( \Gamma \), then the way is open to a solution of Dirichlet's problem. For if we do the computation leading to (4) again, but now with \(|x - y|^{-1}\) replaced by \(G(x, y)\), the last integral of (4) vanishes, and if \( u \) is harmonic we get

\[
u(x) = \int G(x, y) u(y) d\Gamma(y),
\]

(5)