The Shift Equivalence Problem

Subshifts of finite type have appeared naturally in subjects ranging from dynamical systems, to ergodic theory, to statistical mechanics, to C*-algebras, to coding and information theory. A key method in studying them uses strong shift equivalence theory. This grew out of R.F. Williams’s fundamental work [Wi1], in which he formulated an algebraic approach to the topological classification problem for subshifts of finite type by introducing the concepts of strong shift equivalence and shift equivalence, the latter being much more accessible. Since 1974, the question of whether shift equivalence implies strong shift equivalence has been a well-known and tantalizing problem, called the Shift Equivalence Problem or Williams’s Conjecture.

This is a marvelous problem for at least three reasons. First, it comes from the fundamental and still open conjugacy problem in dynamics. Second, it is easy to state in very elementary terms. Third, it is unexpectedly and closely related to other branches of mathematics outside dynamics, such as algebraic K-theory and topological quantum field theory. We will give an account of the Conjecture, its recent disproof, and where this leaves the subject.

The Shift Equivalence Problem came from the 1960s and early 1970s, which were tremendously exciting years for dynamics. Simultaneously, there was an explosion of activity in differential topology and algebraic K-theory. Taken together, the papers [Fr, Mi2, S1] established a close connection between zeta functions, which count periodic points of dynamical systems, and Alexander–Reidemeister torsion, which gives invariants of knots in $R^3$ and is related to the algebraic K-theory group $K_1$. In [C, HW] a connection is made between diffeomorphisms of manifolds and the algebraic K-theory group $K_2$. So it was natural to wonder whether there might be a connection between symmetry groups of dynamical systems and $K_2$. This theme was developed during the 1980s and 1990s, and it ultimately produced machinery which was used in [KR5, KR6] to find the first counterexamples to the Shift Equivalence Problem for primitive matrices in 1997. We will explain how a certain cohomology class $sg_{c_2}$, constructed by studying how symmetries of dynamical systems act on periodic points, is crucial for the counterexamples. Another approach was subsequently found in [W7] using an invariant $\phi_{2m}$ in the algebraic K-theory group $K_2$ of a truncated polynomial ring. The construction of $\phi_{2m}$ comes very directly from the analogy with $K_2$ invariants for diffeomorphisms and is quite dif-
different from the construction of sgnc. The first two authors have recently shown that, surprisingly, \( \mathcal{D}_2 = \text{sgnc} \).

Even though the work on shift equivalence and the work on diffeomorphisms and \( \mathcal{K}_2 \) were done around the same time in the early 1970s by people who were friends, a connection was not made between the two areas at that time. Only now, almost a quarter of a century later, these ideas turn out to be closely related.

This article primarily discusses the approach to the counterexamples using sgnc. See [G, W8] for more details on the relationship among strong shift equivalence, algebraic K-theory, and topological quantum field theory.

The next section explains subclasses of finite type and gives the precise statement of the Shift Equivalence Problem. The following two sections develop a general strategy for analyzing the difference between shift equivalence and strong shift equivalence. Then, we introduce sgnc and proceed to give an explicit counterexample to the Shift Equivalence Problem using it. The final section briefly discusses further problems and topics.

Good general background references are [K, LM, R]. A sampling of articles is [A, BH2, CK2, E, M, S1, S2].

**Subshifts of Finite Type and the Classification Problem**

Messages can be represented as streams of symbols moving from right to left, and it is reasonable to consider two long sequences of symbols to be close to one another or almost the same if they agree except possibly for a few symbols at the beginning and at the end.

This can be made precise by defining the full \( n \)-shift \( X_n \) to be the set of biinfinite sequences \( x = [x_k] \), where each symbol \( x_k \) is drawn from an \( n \)-letter alphabet \( \{0, \ldots, n-1\} \). It is equipped with the product topology making it a Cantor set. The shift homeomorphism \( \sigma_n : X_n \to X_n \) is defined by \( \sigma_n(x)_k = x_{k+1} \) (i.e., shift one space to the left).

A subshift of finite type arises by fixing a finite set \( F \) of words (of finite length) in the symbols \( \{0, \ldots, n-1\} \) and then excluding from \( X_n \) all \( x \) which contain a word in \( F \). Very practical examples are the run-length-limited subshifts used in data storage. Consider the subshift \( L(2, 7) \) of \( X_2 \), which consists of all those \( x \) which have infinitely many ones to the right and left and which have at least two but not more than seven zeros between any two successive ones. This model a magnetic tape zipping by, with 1 being the symbol for reversal of magnetic fields and 0 being the symbol for nonreversal. Insisting upon at least two zeros helps prevent intersymbol interference or confusion of magnetic fields. The bound of seven zeros helps maintain accuracy of a clock, which is updated at each reversal.

Alternatively, and equivalently, we can obtain a subshift of finite type by considering an \( m \times m \) zero-one transition matrix \( A \). Given such an \( A \), define [LM] the subshift of finite type \( [X_A, \sigma_A] \) of the Bernoulli \( m \)-shift \( (X_m, \sigma_m) \) by letting \( X_A \) be the set of sequences satisfying \( A(x_k, x_{k+1}) = 1 \) for all \( -\infty < k < \infty \). In other words, \( X_A \) is obtained by excluding those \( x \) in \( X_m \) that contain 2-blocks \([j]\) where \( A(i,j) = 0 \). By definition, \( \sigma_A = \sigma_m|_{X_A} \).

**Example:** The \( m \times m \) matrix of all 1’s gives the full \( m \)-shift (i.e., any symbol is allowed to follow any other one). The finite set of excluded blocks is empty. Another example is the golden mean shift \( X_{gm} \), which is a subshift of \( X_2 \) and arises from the matrix

\[
A_{gm} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Since \( A_{gm}(1,1) = 0 \), the symbol 1 cannot be followed by itself. Thus, \( X_{gm} \) consists of all sequences of 0’s and 1’s where each 1 must have a 0 to its immediate right. The only excluded block is [11].

The transition matrix is always in \( ZO \), the set of zero—one matrices. Here is another procedure, called the edge shift construction [LM], which yields a subshift of finite type \( (X_A, \sigma_A) \) from any \( m \times m \) non-negative integral matrix \( A = [A_{ij}] \). The matrix \( A \) can be viewed as a directed graph which has \( m \) vertices and which has \( A_{ij} \) edges going from the vertex \( i \) to the vertex \( j \). Order the set \( S^\alpha \) of edges of the graph \( A \), and define the zero—one matrix \( A^\alpha : S^\alpha \times S^\alpha \to [0,1] \) by letting \( A^\alpha(\alpha, \beta) = 1 \) iff the end vertex of the edge \( \alpha \) is the start vertex of the edge \( \beta \). Then, one defines

\[
(X_A, \sigma_A) = (X_{A^\alpha}, \sigma_{A^\alpha}).
\]

This is a subshift of the full shift \( (X_m, \sigma_m) \), where \( m^\# \) is the number of edges of \( A \). If \( A \) is a zero—one matrix, there are two constructions of a subshift of finite type associated to \( A \); namely \( [X_A, \sigma_A] \) and \( (X_A, \sigma_A) = [X_{A^\alpha}, \sigma_{A^\alpha}] \). These are well known to be canonically equivalent. See [W4].

A subshift of finite type is an example of a discrete-time dynamical system \( (X, f) \) (i.e., a homeomorphism \( f : X \to X \) of, say, a compact space \( X \)). For any \( (X, f) \) one seeks its intrinsic properties or invariants; this is made precise by the idea of topological conjugacy. We say that \( f : X \to X \) and \( g : Y \to Y \) are topologically conjugate provided there is a homeomorphism \( \phi : X \to Y \) such that \( \phi \circ f = g \circ \phi \).

One of the most important and basic invariants of topological conjugacy is the topological entropy \( h(f) \) defined by Adler, et al. [AKM]. This is a topological version of entropy in ergodic theory [Kol] and channel capacity in information theory [Sh]. In our case, \( X = X_A \) and \( f = \sigma_A \). The topological entropy can be computed by the beautiful formula

\[
h(\sigma_A) = \lim_{n \to \infty} \frac{1}{n} \log B_n = \log \lambda_A,
\]

where \( \lambda_A \) is the largest eigenvalue of \( A \) and \( B_n \) is the number of distinct \( n \)-letter subwords in a sequence of length \( n \) in \( X_A \).