THE CLASSIFICATION OF 3-DIMENSIONAL
LOCALLY STRONGLY CONVEX
HOMOGENEOUS AFFINE HYPERSURFACES

FRANKI DILLEN AND LUC VRANCKEN

1. INTRODUCTION

In this paper, we will give a complete classification of the 3-dimensional homogeneous, locally strongly convex affine hypersurfaces. We use the same notation as in [DV2]; for a detailed treatment, see [N].

From now on, $M$ will always be a 3-dimensional, locally strongly convex locally homogeneous hypersurface of $\mathbb{R}^4$. Then the shape operator $S$ is always diagonalizable, and the eigenvalues of $S$ are constant along $M$. According to the number of different eigenvalues, we have the following results.

**Theorem 1.** There are no 3-dimensional locally strongly convex, locally homogeneous hypersurfaces in $\mathbb{R}^4$ whose affine shape operator $S$ has three distinct eigenvalues.

We will prove Theorem 1 in Section 2. Next, if $S$ has two distinct eigenvalues, we can use the classification in [DV2] and [DV4] to obtain the following theorem.

**Theorem 2.** Let $M^3$ be a locally strongly convex, locally homogeneous hypersurface in $\mathbb{R}^4$, whose shape operator has two distinct eigenvalues. Then $M$ is affine equivalent to the convex part of one of the following hypersurfaces:

\[
\begin{align*}
(y - \frac{1}{2}(x^2 + z^2))^4w^2 &= 1, \\
(y - \frac{1}{2}x^2)^3(z - \frac{1}{2}w^2)^3 &= 1, \\
(y - \frac{1}{2}x^2)^3u^2w^2 &= 1, \\
(y - \frac{1}{2}x^2 - \frac{1}{2}w^2)^4z^3 &= 1,
\end{align*}
\]

where $(x, y, z, w)$ are the coordinates of $\mathbb{R}^4$.

Finally, if $S$ is a multiple of the identity, $M$ is called an affine sphere. Globally homogeneous affine spheres have been classified in [S] by T. Sasaki, see also [DV3]. In Section 3 and Section 4, we will obtain a classification of the locally homogeneous affine hyperspheres. First, in Section 3, we will construct a special differentiable frame which we will use in Section 4 to prove the following theorem.

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Both authors are Senior Research Assistant of the National Fund for Scientific Research (Belgium)
Theorem 3. Let $M$ be a 3-dimensional locally strongly convex, locally homogeneous hypersphere in $\mathbb{R}^4$. Then either $M$ is an open part of a locally strongly convex quadric or $M$ is affine equivalent to an open part of one of the following two hypersurfaces:

1. $xyzw = 1$,
2. $(y^2 - z^2 - w^2)\lambda^2 = 1$.

Clearly the above three theorems together give a complete classification of the 3-dimensional, locally homogeneous, locally strongly convex affine hypersurfaces in $\mathbb{R}^4$.

2. Proof of Theorem 1

Let $p \in M$. Since $M$ is locally strongly convex and $S$ has 3 different eigenvalues, we can take $h$-orthonormal tangent vector fields $E_1$, $E_2$, $E_3$ on a neighborhood of $p$ such that

\[
SE_1 = \lambda_1 E_1 \\
SE_2 = \lambda_2 E_2 \\
SE_3 = \lambda_3 E_3.
\]

We introduce functions $\gamma_{ij}^k$ by

\[
\nabla_{E_i} E_j = \sum_{k=1}^{3} \gamma_{ij}^k E_k.
\]

Since at every point the vector fields $E_1$, $E_2$ and $E_3$ are uniquely determined, the functions $\gamma_{ij}^k$ are constant.

Lemma 2.1. For $i \neq j$, we have that $\gamma_{ij} = 0$.

Proof. From the equation of Codazzi for $S$, we obtain

\[
0 = (\nabla_{E_i} S) E_j - (\nabla_{E_j} S) E_i \\
= \lambda_j \nabla_{E_i} E_j - S(\nabla_{E_i} E_j) - \lambda_i \nabla_{E_j} E_i + S(\nabla_{E_j} E_i).
\]

Taking then the $E_i$ and the $E_j$ component of this equation gives us that

\[
\gamma_{ij}^i = 0 = \gamma_{ji}^j.
\]

Lemma 2.2. Let $i, j, k$ be 3 mutually different numbers, then

\[
(\lambda_j - \lambda_k)\gamma_{ij}^k = (\lambda_i - \lambda_k)\gamma_{ji}^k.
\]

Proof. We look at (2.1). Take $k$ different from both $i$ and $j$ and consider the component in the direction of $E_k$. From this, the proof follows immediately.