A THEOREM ON SUMS OF POWERS WITH APPLICATIONS TO THE ADDITIVE THEORY OF NUMBERS.

By S. CHOWLA,
Andhra University, Waltair.

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1. Let us denote by $N(k)$ the least value of $s$ such that the equation

$$\sum_{m \leq s} a_m^t = \sum_{n \leq s} b_n^t \quad (1 \leq t \leq k)$$

has a solution in integers $a_m (m \leq s)$ and $b_n (n \leq s)$ where

$$a_m (m \leq s) \neq b_n (n \leq s).$$

We shall show that

Theorem.

$$N(k) \leq \frac{k^2 + k}{2} + 1.$$  

This is an improvement on previous results. 1

2. In what follows the $B$'s are real numbers which vary from place to place but are greater than a positive constant depending only on $k$ and $s$. $n_1, \ldots, n_s$ denote positive integers.

It follows from Dirichlet's principle ("Schubfachschluss") that there infinitely many integers $n$ with the property that the equation

$$n_1^k + \cdots + n_s^k = n$$

has $Bn^k - 1$ solutions in $n_1, \ldots, n_s$.

Now (4) implies

$$n_1 + \cdots + n_s \leq \frac{1}{sn^{-k}}.$$  

From (4), (5) and the "Schubfachschluss" it follows that there are infinitely many sets $n, n'$ of integers with the property that the equations

$$n = \sum_{r \leq s} n_r^k$$

$$n' = \sum_{r \leq s} n_r$$

have $Bn^{-k} - 1 - \frac{1}{k}$ solutions in $n_r (r \leq s)$.


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Again (6) implies

\[(7) \quad \sum_{r \leq s} n_r^2 \leq s n^k.\]

Hence ("Schubfachschluss") from (6) and (7) it follows that there are infinitely many sets of integers \((n, n', n'')\) with the property that the equations

\[
\begin{align*}
\begin{cases}
n = \sum_{r \leq s} n_r^k, \\
n' = \sum_{r \leq s} n_r, \\
n'' = \sum_{r \leq s} n_r^2 
\end{cases}
\]

have \(Bn^k - 1 - \frac{1}{k} - \frac{2}{k^2}\) solutions in \(n_r (r \leq s)\). Proceeding in this manner we arrive at the result that there are infinitely many sets of integers \(n, n', n'', \ldots, n^{(k-2)}, n^{(k-1)}\) such that the system of equations

\[
\begin{align*}
\begin{cases}
n = \sum_{r \leq s} n_r^k, \\
n^{(m)} = \sum_{r \leq s} n_r^m \quad (1 \leq m \leq k - 1) 
\end{cases}
\]

has \(Bn^k\) solutions where

\[
(10) \quad \phi = \frac{s}{k} - 1 - \left(\frac{1}{k} + \frac{2}{k} + \cdots + \frac{k-1}{k}\right).
\]

Now

\[
(11) \quad \phi > 0 \text{ if } s \geq \frac{k^2 + k}{2} + 1.
\]

Hence our theorem.

3. Now (1) implies

\[
(12) \quad \sum_{a} (x + a)^k = \sum_{b} (x + b)^k.
\]

Hence our theorem shows that if \(s \geq \frac{k^2 + k}{2} + 1\), we can find two different sets of integers \(a_1, \ldots, a_s\) and \(b_1, \ldots, b_s\) such that

\[
(13) \quad \sum_{m \leq s} (x + a_m)^k = \sum_{n \leq s} (x + b_n)^k.
\]

It follows immediately that

\[
(14) \quad \beta(k) \leq \frac{k^2 + k}{2} + 1
\]

here \(\beta(k)\) is the function defined by Steen and Rao\(^2\).