SOME PROPERTIES OF THE $k$-FUNCTION WITH
NON-INTEGRAL INDEX.

BY N. A. SHASTRI, M.Sc.,
Department of Mathematics, College of Science, Nagpur.

Received March 27, 1935.

I. THE contour integral for $k_{2n}(x)$ is

$$k_{2n}(x) = -\frac{e^{-x}}{2\pi i n} \int_{0+}^{\infty} e^{-t}(-t)^{-n} (2x + t)^n \, dt \quad \ldots \quad (1.1)$$

where the integrand is rendered one-valued by taking $|\arg(-t)| \leq \pi$, and the contour is so chosen that the point $t = -2x$ lies outside it. Changing $t$ into $-u$ in (1.1) we get

$$k_{2n}(x) = \frac{e^{-x}}{2\pi i n} \int_{\infty}^{0+} e^u u^{-n} (2x - u)^n \, du \quad \ldots \quad (1.2)$$

Let the contour in (1.2) consist of the real axis from $(0$ to $-\infty$) taken twice and a small circle enclosing the origin. The integral along this circle vanishes when $R(n) < 1$. It can be easily shown that for such a contour

$$k_{2n}(x) = \frac{\sin n\pi}{n\pi} \int_{0}^{\infty} e^x \rho^{-n} (\rho + 2x)^n \, d\rho$$

where $R(n) < 1$ and $n$ not an integer. This integral holds for all values of $x$ real or complex except negative real values. Throughout this paper it will be assumed that $n$ is non-integral and $R(n) < 1$ and $x$ takes any value real or complex except any negative real value.

II. Addition Theorem.

From (1.3) with the restrictions on $n$ and $x$, we have

$$k_{2n}(x + y) = 2(x+y) \sin \frac{n\pi}{n\pi} \int_{-1}^{1} e^{x \frac{t-1}{t+1}} e^{y \frac{t-1}{t+1}} (-t)^{-n} \frac{dt}{(1+t)^2} \left[ \frac{x-u}{x+u} = t \right] \quad \ldots \quad (2.1)$$

Hence

$$k_{2n}(x+y) = 2(x+y) \sin \frac{n\pi}{n\pi} \int_{-1}^{1} e^{x \frac{t-1}{t+1}} e^{y \frac{t-1}{t+1}} (-t)^{-n} \frac{dt}{(1+t)^2} \left[ \sum_{r=0}^{\infty} k_{2r}(y)t^r \right] \frac{(-t)^{-n}}{(1+t)^2} \, dt$$

using the expression for the generating function\(^1\) of the $k$-function.

\(^1\) Bateman, Trans. Amer. Math. Soc., 33, 817-831 (2.2).

768
Some Properties of \( k \)-Function with Non-Integral Index

Hence

\[
k_{2n}(x+y) = \left(1 + \frac{y}{x}\right) \sum_{r=0}^{\infty} k_{2r}(y) \left(-\right)^{r} \frac{2x \sin n\pi}{n\pi} \int_{-1}^{0} e^{x \left(\frac{t-1}{t+1}\right)} \left(-\right)^{r} \frac{(1+t)^{r-n}}{(1+t)^{2}} \, dt
\]

\[
= \left(1 + \frac{y}{x}\right) \sum_{r=0}^{\infty} \frac{\left(-\right)^{r} (n-r)\pi \sin n\pi}{n\pi \sin (n-r)\pi} k_{2n-2r}(x) k_{2r}(y).
\]

Therefore

\[
n k_{2n}(x+y) = \left(1 + \frac{y}{x}\right) \sum_{r=0}^{\infty} (n-r) k_{2n-2r}(x) k_{2r}(y).
\] .. .. .. (2·3)

If \( x = y \)

\[
n k_{2n}(2x) = \sum_{r=0}^{\infty} (2n-2r) k_{2n-2r}(x) k_{2r}(x).
\] .. .. .. (2·4)

The term by term integration in (2·2) can be justified by using a theorem similar to the theorem in §70·2, Carslaw—Theory of Fourier's Series and Integrals (1930).

Expansions.

III. We have from (2·1) with the usual restrictions on \( n \) and \( x \)

\[
k_{2n}(x) = \frac{2x \sin n\pi}{n\pi} \int_{-1}^{0} e^{x \frac{t-1}{t+1}} \frac{\left(-\right)^{r} n}{(1+t)^{2}} \, dt
\]

\[
= \frac{2x \sin n\pi}{n\pi} \int_{0}^{\infty} e^{-x-2xt'} \left(1 + \frac{1}{t'}\right)^{n} \, dt' \quad \left[t' = \frac{-t}{1+t}\right] \quad .. (3·1)
\]

Hence

\[
k_{2n}\left(\frac{x}{u}\right) = \frac{2x \sin n\pi}{n\pi u} \int_{0}^{\infty} e^{-\frac{x}{u}} \left(1 + \frac{1}{t'}\right)^{n} \, dt'
\]

\[
= \frac{2x \sin n\pi}{n\pi} \int_{0}^{\infty} e^{-\frac{x}{u}} \frac{1}{u} - 2xv (\frac{1}{v} + u)^{n} u^{-n} \, dv \quad \text{[using } t' = uv \text{, } u \text{ positive]}
\]

\[
= \frac{2x \sin n\pi}{n\pi} \int_{0}^{\infty} e^{-\frac{x}{u}} - 2xv u^{-n} (1 + \frac{1}{v})^{n} \left[1 + \frac{u - \frac{1}{v}}{1 + \frac{1}{v}}\right]^{n} \, dv
\]