ON THE INVERSES OF A CIRCLE WITH RESPECT TO A TETRAD OF FIXED CIRCLES AND THEIR ORTHOGONAL TETRAD

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1. Let $C_i$ ($i = 1, 2, 3, 4$) be four circles and $S_i$ the four circles respectively orthogonal to sets of three chosen from $C_i$. The main object of this paper is then to establish the following results.

If the inverses of a point $P$ w.r.t. the circles $C_i$ lie on a circle, then the inverses of $P$ w.r.t. $S_i$ also lie on a circle. The locus of such points $P$ is an octavic curve $F_1$ having quadruple points at the circular points at infinity $(1.1)$

The totality of circles $\Sigma$ whose inverses w.r.t. $C_i$ have a common orthogonal circle as also the inverses w.r.t. $S_i$ consists of the four coaxal systems respectively conjugate to the four systems defined by the pairs $C_i, S_i$; ($i = 1, 2, 3, 4$) and a family of circles whose centres lie on a quartic curve. $(1.2)$

If the inverses of a circle $\Sigma$, w.r.t. the circle $C_i$ have a common orthogonal circle $\Sigma'$, the transformation in circle-space carrying $\Sigma$ to $\Sigma$ is the involutory cubic transformation whose singular points are those representing the circles $S_i$ and whose fixed points represent the circles cutting the circles $C_i$ at equal angles. $(1.3)$

Lastly the following theorem relating to the Miquel-Clifford configuration is proved.

If the inverses of a point $P$ w.r.t. $n$ concurrent circles $C_i$ lie on a circle then the inverses of $P$ w.r.t. every concurrent set of $n$ circles of the Miquel-Clifford configuration generated by the circles $C_i$ also lie on a circle. $(1.4)$

2. It is well known that the $\infty^3$ circles of a plane $\pi$ may be represented by the points of a projective space $S_3$, the $\infty^2$ point circles corresponding to points on a quadric $Q$ called the Absolute. Let us represent, for convenience, by the same symbol both the circle on $\pi$ and its corresponding point in $S_3$. Let $\Delta_1, \Delta_2$ be the two tetrahedra whose vertices represent $C_i$ and $S_i$ so

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that \( \Delta_1, \Delta_2 \) are reciprocals of each other in regard to \( \Omega \). If the inverses of a point \( P \) on \( \pi \) w.r.t. the circles \( C_i \) are concyclic, it is easy to see that in \( S_9 \) the projections of \( C_i \) on \( \Omega \) from \( P \) as vertex of projection are coplanar. In this case, the generators \( g_1, g_2 \) of \( \Omega \) at \( P \) and the lines joining \( P \) to the vertices \( C_i \) of \( \Delta_1 \) all lie on a quadric cone of vertex \( P \) and hence \( g_1 \) and \( g_2 \) both belong to the same tetrahedral complex \( \Gamma \) whose fundamental tetrahedron is \( \Delta_1 \). Hence \( g_1, g_2 \) and the four lines of intersection of the faces of \( \Delta_1 \) with the tangent plane \( p \) to \( \Omega \) at \( P \) all touch a conic, \( \text{viz.} \), the complex conic of \( \Gamma \) in the plane \( p \). Reciprocating this result in regard to \( \Omega \) we immediately see that \( g_1, g_2 \) and the lines joining \( P \) to the vertices \( S_i \) of \( \Delta_2 \) all lie on a quadric cone of vertex \( P \). Hence the projections of \( S_i \) from \( P \) on \( \Omega \) are coplanar. Hence on \( \pi \) the inverses of \( P \) w.r.t. the circles \( S_i \) lie on a circle. Thus the first part of (1.1) is proved. As a particular case of (1.1), we have the theorem that if the centres of four circles \( C_i \) lie on a circle then the centres of the four circles \( S_i \) respectively orthogonal to sets of three chosen from \( C_i \) also lie on a circle.

3. Next, taking \( \Delta_1 \) as the tetrahedron of reference let the homogeneous co-ordinates of points in \( S_9 \) be so chosen that the equation of the Absolute takes the form

\[
\Omega = a_{11} x_1^2 + \cdots + a_{44} x_4^2 + 2a_{12} x_1 x_2 + \cdots = 0.
\]

If two circles \( \Sigma, \Sigma' \) are inverses of one another in regard to a circle \( C \) it is known that in \( S_9 \), \( \Sigma, \Sigma' \) are collinear with \( C \) and separate harmonically \( C \) and the point of intersection of the line with the polar plane of \( C \) in regard to \( \Omega \). The use of this property shows that if \( X \) be a circle of co-ordinates \( x_i \) and \( X_i \) the four circles which are respectively the inverses of \( X \) w.r.t. \( C_i \), then the co-ordinates of \( X_i \) are obtained from those of \( X \) by simply changing \( x_i \) into \( x_i - \frac{1}{a_{ii}} \frac{\partial \Omega}{\partial x_i} \) and leaving the three other co-ordinates unaltered. The condition of coplanarity of the points \( X_i \) is then easily seen to be

\[
\psi_i \equiv \frac{a_{11} x_1}{\hat{p}_i} + \cdots + \frac{a_{44} x_4}{\hat{p}_i} - 2 = 0 \tag{3.1}
\]

where

\[
\hat{p}_i = \frac{1}{2} \frac{\partial \Omega}{\partial x_i} \quad (i = 1, 2, 3, 4).
\]

Hence the \( \infty^2 \) circles \( C \) on \( \pi \) which are such that the inverses of \( C \) w.r.t. \( C_i \) have a common orthogonal circle are represented in \( S_9 \) by the points of the quartic surface \( \psi_1 \). The surfaces \( \psi_1 \) and \( \Omega \) intersect in an octavic curve \( \Gamma_1 \). From the definitions of \( \psi_1 \) and \( \Omega \) it is evident that the points of \( \Gamma_1 \) represent points \( P \) on \( \pi \) which are such that the inverses of \( P \) w.r.t. the circles \( C_i \) lie