ON LINEAR TRANSFORMATIONS OF BOUNDED SEQUENCES—III.

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Received August 4, 1938.

PART III.

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This part deals with a subclass of T [T as defined in 2·1 of Part I of this paper]. We designate the direct and inverse transformations of this class by U and U⁻¹, and prove that these transformations, and others defined by their products are commutative. We further show that transformations corresponding to differences of any real order form a subclass of the group defined by U, U⁻¹, and their products. In (16) we show that some important theorems of Anderson (A. 1)* are, either deducible from, or particular cases of, theorems of Parts I and II of this paper. In (17) we discuss the generalization of Knopp's results on "Mehrfach monotone folgen" (K. 2).†


Let ||aₘₙ|| define a U. Then besides the four conditions of (2·1) of Part I, condition (e), namely, aₙₙ₊ᵣ = aᵣ for all n, i.e., a₀ᵣ = a₁ᵣ₊₁ = a₂ᵣ₊₂ = ... , characterizes aₘₙ; so that aₘₙ for U is characterized as follows:

(a') aₙₙ = 1   (b') aₘₙ = 0, n < m   (c') aᵣ < 0 for all r ≥ 1

[14·1]

(d') $\sum_{r=1}^{\infty} aᵣ \leq 1$. We see that a U is defined completely by a sequence {aᵣ} satisfying (c') and (d').

(A·1)*. A. F. Anderson Studier over Cesaro's summabilitets methode (Danish). See the second chapter entitled "Om differencer".

In section 2 of Part I we established the existence of a unique reciprocal matrix \( \| \beta_{m,n} \| \) such that \( \| \beta_{m,n} \| \cdot \| a_{m,n} \| = \| \delta_{m,n} \| \) (unit matrix). Since any \( U \) is a \( T \) it follows that in this case also \( \| \beta_{m,n} \| \) the reciprocal matrix exists.

Further from section 2 of Part I, we obtain

\[
\beta_{n,n+p} = - \sum_{k=1}^{p} a_{n,n+k} \beta_{n+k,n+p} \quad (2.6)
\]

We can at once deduce \( \beta_{n,n+p} = b_{p} \) for all \( n \); and

\[
b_{p} = - \sum_{k=1}^{p} a_{n} b_{p-k}. \quad [14.2]
\]

We obtain the following results also easily. \( b_{0} = 1 \), \( b_{n} > 0 \), and from (14.2) it follows that \( b_{n} \) is given by the equation

\[
\left( \sum_{n=0}^{\infty} b_{n} x^{n} \right) \cdot \left( 1 + \sum_{n=1}^{\infty} a_{n} x^{n} \right) = 1. \quad [14.3]
\]

If \( U_{1} \) and \( U_{2} \) are defined by \( \{ a_{n} \} \) and \( \{ a_{n}^{2} \} \), it is easy to prove

(1) \( U_{1} U_{2} = U_{2} U_{1} \); (2) if \( \| c_{m,n} \| \) defines \( U_{1} U_{2} \) then \( c_{n,n+p} = a_{p}^{2} \) for all \( n \),

(3) \( a_{p}^{3} \) is given by the equation

\[
1 + \sum_{p=1}^{\infty} a_{p}^{3} x^{p} = \left( 1 + \sum_{p=1}^{\infty} a_{p}^{1} x^{p} \right) \left( 1 + \sum_{p=1}^{\infty} a_{p}^{2} x^{p} \right). \quad [14.4]
\]

The matrix of any product of \( U \)'s and \( U^{-1} \)'s is always characterized by condition (e) of 14.1. If \( \{ a_{p} \} \) defines the product \( a_{p} \) can be calculated in all cases from an equation of the type of (14.4). It is quite easy to shew that the commutative property is true for any product of \( U \)'s and \( U^{-1} \)'s.

§ 15. Differences of any Real Order.

**Theorem**: Transformations defined by differences of any real order form a subclass of the class formed by \( U \)'s, \( U^{-1} \)'s, and their products.

**Lemma 1.** If \( 0 < \gamma < 1 \) then we shall prove that \( \Delta^{\gamma} U = U (\gamma) \).

Formally the difference \( \Delta^{\gamma} v_{n} = v_{n} - \gamma v_{n+1} + \gamma (\gamma - 1) \frac{v_{n+2}}{2} - \gamma (\gamma - 1) (\gamma - 2) \frac{v_{n+3}}{3} + \cdots \)

\[
= v_{n} - \gamma v_{n+1} - \gamma (1 - \gamma) \frac{v_{n+2}}{2} - \gamma (1 - \gamma) \frac{v_{n+3}}{3} - \cdots
\]

\[
= v_{n} - \gamma v_{n+1} - \gamma (1 - \gamma) \cdots (\phi - 1 - \gamma) \frac{v_{n+p}}{p}.
\]

Consider a transformation \( U (\gamma) \) defined by \( \{ a_{n} \} \) as follows:

\[
a_{1} = - \gamma a_{2} = - \gamma (1 - \gamma) \cdots a_{p} = - \gamma (1 - \gamma) (\phi - 1 - \gamma)
\]

then, \( a_{p} < 0 \) for \( \phi \geq 1 \) and \( - \sum_{1}^{\infty} a_{p} = 1. \)