SERIES INVOLVING PRODUCTS OF LAGUERRE POLYNOMIALS

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In this note the sums of two interesting series involving products of Laguerre polynomials are given. The heuristic proofs given here use the method of Laplace transforms.*

It will now be shown that

\[
\sum \frac{n!}{\Gamma(a+n+1)} L_n^\alpha(x) L_n^\beta(y) z^n = \exp\left[\frac{-(x+y)z}{1-z}\right](1-z)^{-(\beta+1)}
\]

\[\times \Phi_3\left[\begin{array}{c}
\alpha - \beta, \ a + 1, \ \frac{xz}{1-z}, \ \frac{xyz}{(1-z)^2}
\end{array}\right] \tag{1}\]

where

\[
\Phi_3(\beta, \gamma, x, y) = \sum_{m,n} \frac{(-\beta)_m}{(-\gamma)_{m+n}} \frac{x^m y^n}{m! n!}
\] \tag{2}

is the hypergeometric function of two arguments (cf. Erdelyi,\(^1\) p. 225).

We rewrite the familiar result

\[
\sum L_n^\beta(y) s^n = (1-s)^{-(\beta+1)} \exp\left[\frac{ys}{s-1}\right] \tag{3}\]

* In principle, this method is the same as that used by Varma (Refer Proc. Ind. Acad. Sci., 12 A, 52).


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(cf. Erdelyi,² p. 189) as,

\[
\frac{1}{(p-x)^{a+1}} \sum_{n=0}^{\infty} L_n^\beta (y) z^n \left(1 - \frac{x}{p-x}\right)^n
\]

\[= \exp \left[ -\frac{yz}{(1-z)} \right] \exp \left( \frac{Y}{p} \right) (1 - z)^{-(\beta+1)}
\]

\[\times \left[ \frac{p}{(p-x)} \right]^{-(\beta+1)} \left[ \frac{1}{(p-x)} \right]^{a+1}; \]

\[ (4) \]

Here

\[Y = \frac{xyz}{(1-z)^2} \quad \text{and} \quad X = \frac{xz}{1-z}.\]

We use \(\mathcal{L}\{f(t)\} = \tilde{f}(p)\) to denote the Laplace transform of \(f(t)\) \((p\text{-transform variable})\) so that \(\mathcal{L}^{-1}\{\tilde{f}(p)\} = f(t)\).

Also,

\[\mathcal{L}\{t^n \exp (xt) L^a_n (xt)\}\]

\[= \frac{\Gamma(a + n + 1)}{n!} \frac{(p - x - X)^n}{(p - x)^{a+n+1}} \quad \text{Re} (p - X) > 0. \]

\[ (5) \]

(cf. Erdelyi,³ p. 175)

and

\[\mathcal{L}\{t^a \Phi_a (\alpha - \beta, \alpha + 1, xt, yt)\}\]

\[= \Gamma(\alpha + 1) p^{-(\alpha+1)} \left(1 - \frac{X}{p}\right)^{\beta-a}\]

\[= \Gamma(\alpha + 1) \left(1 - \frac{X}{p}\right)^{\beta+1} (p - X)^{-(\alpha+1)} \exp \left( \frac{Y}{p} \right) \]

\[\text{Re} p > 0, X \text{ real} \]

\[ (6) \]

(cf. Erdelyi,³ p. 223)