SERIES INVOLVING PRODUCTS OF
LAGUERRE POLYNOMIALS

BY S. K. RANGARAJAN
(Central Electro-chemical Research Institute, Karaikudi)

Received July 17, 1963
(Communicated by Dr. K. S. G. Doss, F.A.S.C.)

In this note the sums of two interesting series involving products of Laguerre polynomials are given. The heuristic proofs given here use the method of Laplace transforms.*

It will now be shown that

\[
\sum \frac{n!}{\Gamma(\alpha + n + 1)} L_n^a(x) L_n^\beta(y) z^n = \exp \left[ \frac{- (x + y) z}{(1 - z)} \right] (1 - z)^{(\beta + 1)}
\]

\[\times \Phi_3 \left[ a - \beta, \ a + 1, \ \frac{xz}{1 - z}, \ \frac{xyz}{(1 - z)^2} \right] \tag{1}\]

where

\[
\Phi_3(\beta, \gamma, x, y) = \sum_{m, n} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!} \tag{2}
\]

is the hypergeometric function of two arguments (cf. Erdelyi,\(^1\) p. 225).

We rewrite the familiar result

\[
\sum L_n^\beta(y) s^n = (1 - s)^{-(\beta + 1)} \exp \left[ \frac{ys}{s - 1} \right] \tag{3}\]

* In principle, this method is the same as that used by Varma (Refer Proc. Ind. Acad. Sci., 12 A, 52).
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(cf. Erdelyi, p. 189) as,

\[
\frac{1}{(p - X)^{\alpha+1}} \sum_{n=0}^{\infty} L_n^\beta(y) \left(1 - \frac{x}{p - X}\right)^n
\]

\[= \exp\left[\frac{-yz}{(1 - z)}\right] \exp\left(\frac{Y}{p}\right)(1 - z)^{-(\beta + 1)} \]  

\[
\times \left[\frac{p}{(p - X)}\right]^{-(\beta + 1)} \left[\frac{1}{(p - X)}\right]^{\alpha+1};
\]

(4)

Here

\[Y = \frac{x y z}{(1 - z)^2} \quad \text{and} \quad X = \frac{x z}{1 - z}.
\]

We use \(L\{f(t)\} = \tilde{f}(p)\) to denote the Laplace transform of \(f(t)\). \((p\)-transform variable) so that \(L^{-1}\{\tilde{f}(p)\} = f(t)\).

Also,

\[
L\{t^\alpha \exp(Xt) L_n^\alpha(xt)\}
\]

\[
= \frac{\Gamma(\alpha + n + 1)}{n!} \left(\frac{p - x - X}{p - x}\right)^n \quad \text{Re} \ (p - X) > 0.
\]

(5)

(cf. Erdelyi, p. 175)

and

\[
L\{t^a \Phi_{\alpha}(\alpha - \beta, \alpha + 1, Xt, Yt)\}
\]

\[
= \Gamma(\alpha + 1) p^{-(\alpha + 1)} \left(1 - \frac{X}{p}\right)^{\beta-a} \]

\[
= \Gamma(\alpha + 1) \left(1 - \frac{X}{p}\right)^{\beta+1} (p - X)^{-(\alpha + 1)} \exp\left(\frac{Y}{p}\right) \]

\[\text{Re} \ p > 0, \ X \text{ real}
\]

(6)

(cf. Erdelyi, p. 223)