ON EXCEPTIONAL VALUES OF ENTIRE AND MEROMORPHIC FUNCTIONS

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ABSTRACT

Let \( f(z) \) be meromorphic function of finite nonzero order \( p \). Assuming certain growth estimates on \( f \) by comparing it with \( r^p L(r) \) where \( L(r) \) is a slowly changing function we have obtained the bounds for the zeros of \( f(z) - g(z) \) where \( g(z) \) is a meromorphic function satisfying \( T(r, g) = o \{ T(r, f) \} \) as \( r \to \infty \). These bounds are satisfied but for some exceptional functions. Examples are given to show that such exceptional functions exist.

1. Let \( f(z) \) be a meromorphic function of order \( \rho (0 < \rho < \infty) \). If \( f(z) \) is an entire function let \( M(r, f) = \max |f(z)| \) on \( |z| = r \). Let \( T(r, f) \) be the Nevanlinna's characteristic function for \( f(z) \) and \( g_1(z), g_2(z), \ldots \) be any set of functions satisfying

\[ T(r, g_i(z)) = o \{ T(r, f) \} \text{ as } r \to \infty (i = 1, 2, \ldots). \] (1.1)

Let \( n(r, x), \tilde{n}(r, x) \) be the number of zeros and the number of distinct zeros respectively of \( f(z) - x \) and \( \tilde{n}(r, f - g) \) the number of distinct zeros of \( f(z) - g(z) \) in \( |z| \leq r \). Define

\[ \tilde{N}(r, \frac{1}{f - g}) = \int_0^r \tilde{n}(t, f - g) \frac{dt}{t}. \]

If \( g \) is an infinite constant let \( \tilde{n}(r, f - g) = \tilde{n}(r, f) \) the number of distinct poles of \( f(z) \) in \( |z| \leq r \).

In this paper we study the exceptional values of the function \( f(z) \) by making use of the comparison function \( r^p L(r) \) where \( L(r) \) is a slowly increasing function satisfying
\[ L(Ct) \sim L(t) \text{ as } t \to \infty \text{ for every fixed positive } C. \] Let \( k \) denote any constant \( \geq 1 \) and
\[
h(\rho) = \left\{ \rho + (1 + \rho^2)^{\frac{1}{2}} \left\{ \frac{1}{\rho} + (1 + \rho^2)^{\frac{1}{2}} \right\} \right\} (\rho > 0). \tag{1.2}
\]
Let \( A \) be a constant not necessarily the same at each occurrence.

**Theorem 1.**—If \( f(z) \) is an entire function of order \( \rho (0 < \rho < \infty) \) satisfying
\[
\log M(kr, f) = a (0 < a \leq \infty) \tag{1.3}
\]
then
\[
\limsup_{r \to \infty} \frac{\log M(kr, f)}{r^\rho L(r)} = a \tag{1.4}
\]
and
\[
\limsup_{r \to \infty} \frac{N(r, f - g)}{r^\rho L(r)} \geq \frac{a}{2k^\rho h(\rho)} \tag{1.5}
\]
for every entire function \( g(z) \) (including a polynomial or a finite constant) satisfying (1.1) with one possible exception.

Remark.—The exceptional function may actually exist. Consider for example
\[
f(z) = \prod_{n=2}^{\infty} \left( 1 + \frac{z}{n (\log n)^2} \right).
\]
Here
\[
\bar{n}(r, 0) \sim \{r/(\log r)^2\}; \quad \log M(r, f) \sim (r/\log r).
\]
Set
\[
r^\rho L(r) = r^\rho (r)
\]
where
\[
\rho(r) = 1 - \frac{\log \log r}{\log r}
\]