ON SOME INFINITE SERIES INVOLVING ARITHMETICAL FUNCTIONS.

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1. Let

\[ \{ x \} = - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin (2n\pi x)}{n} \]

We shall prove the

THEOREM. If \( c_n = O(n^\phi) \) where \( \phi < \frac{1}{2} \), then

\[ \sum_{n=1}^{\infty} \frac{c_n}{n} \{ n\theta \} = - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{G_n \sin (2n\pi \theta)}{n} \]

where \( G_n = \sum_{d|n} c_d \), is true for almost all \( \theta \).

We follow the method used by Davenport to prove the special cases \( c_n = \mu(n), \lambda(n), \Lambda(n) \) which are the well-known arithmetical functions [Quar. J. of Maths. (Oxford), March 1937]. Our theorem also includes the cases \( c_n = \text{no. of divisors of } n \), etc., discussed by Walfisz and the author (Acta Arithmetica, Band I).

The letter D is used as a reference to Davenport’s paper cited above.

2. Let

\[ R_N(\theta) = \sum_{n=1}^{N} \left[ \frac{c_n}{n} \{ n\theta \} + \frac{1}{\pi} \frac{G_n \sin (2n\pi \theta)}{n} \right] \]

Then (D.)

\[ \int_0^1 R_n^2 (\theta) \, d\theta = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{c_m c_n (m, n)^2}{12 m^s n^z} - \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{G_n^2}{n^z} \]

3. We shall show that the expression on the right-hand side above is 0 when \( N = \infty \).

We have the formal identity

\[ \sum_{n=1}^{\infty} \frac{G_n^2}{n^z} = \zeta(s) \times \sum_{n=1}^{\infty} \frac{c_m c_n (m, n)^2}{m^s n^z} \]

where \( \zeta(s) \) is Riemann’s zeta-function. For comparing coefficients of \( n^{-z} \) on both sides, we have

\[ G_n^2 = \sum_{d|n} h(d) \]
where
\[ h_v = \sum_{\mu = 1}^\infty \frac{c_{\mu} c_n}{(m, \mu) = v} \]
which is obvious since
\[ G_n = \sum_{d|n} c_d \]
Putting \( s = 2 \) we have the desired result.

4. It now follows that the double series of Section 2 is equal to the double series
\[
- \sum_{\text{Max} (m, n) > N} \frac{c_m c_n (m, n)^2}{m^2 n^2} = O \left( \sum_{m > N} \frac{c_m c_n (m, n)^2}{m^2 n^2} \right) = O \left( \sum_{m > N} \frac{m^2 \phi + \epsilon}{m^2} \right) = O \left( \frac{N^2 \phi + \epsilon}{N} \right)
\]
since \( \phi < \frac{1}{3} \).

Since
\[ G_n = O (n^\phi + \epsilon) \]
it follows that the right-hand side of the second expression in Section 2 is
\[ O \left( N^{2 \phi - 1} + \epsilon \right) \]

5. We shall now show that \( R_m^2(\theta) \rightarrow 0 \) as \( N \rightarrow \infty \) for almost all \( \theta \). It will be sufficient to prove that
\[ R_m^2(\theta) \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty \]
for almost all \( \theta \), since if \( m^2 \leq n < (m + 1)^3 \) we have
\[
| R_n(\theta) - R_m^2(\theta) | = O \left( \frac{\phi}{m^3} \right) = O \left( \frac{m^2}{m^3} \right) = O \left( \frac{1}{m^2} \right)
\]
which tends to 0 as \( m \rightarrow \infty \) since \( \phi < \frac{1}{3} \).

Let \( E_m \) denote the set of points \( \theta \) at which
\[ | R_m^2(\theta) | > \frac{1}{\log m} \]