GENERALIZATION OF NORMAL CURVATURE OF A CURVE IN A RIEMANNIAN V_n

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1. The object of the present paper is to define an invariant which is a generalization of the expression for the normal curvature of a curve in V_n, and to obtain generalizations of some known results.

2. Consider a Riemannian space V_n of co-ordinates x^i, i = 1, 2, ... n and metric

\[ ds^2 = g_{ij} dx^i dx^j \quad , \quad (2.1) \]

imbedded in a V_{n+1} of co-ordinates y^a, a = 1, 2, ... n + 1 and metric

\[ ds^2 = a_{\alpha\beta} dy^\alpha dy^\beta . \quad (2.2) \]

We have the relation

\[ g_{ij} = a_{\alpha\beta} y^\alpha , i \quad , \quad (2.3) \]

where comma denotes covariant differentiation.

Let N^a denote the contravariant components of the unit normal to V_n, then

\[ a_{\alpha\beta} N^a N^\beta = 1 , \quad (2.4) \]

and

\[ a_{\alpha\beta} N^a y^\beta ; i = 0 , \quad (i = 1, 2, \ldots n) \quad (2.5) \]

where (;) followed by an index denotes generalized covariant derivative (or tensor derivative) with respect to the x with that index.

We have

\[ N^a ; i = - \Omega_{ijk} y^s , k , \quad (2.6) \]

where \( \Omega_{ijk} \) are the components of a symmetric covariant tensor of the second order given by

\[ \Omega_{ij} = y^s ; ij a_{\alpha\beta} N^\beta . \quad (2.7) \]
From (2.6) we have
\[ N_{a;ie} = - \epsilon^i \Omega_{ij} g^{ik} y_{s;k} , \]  
(2.8)
where \( \epsilon^i \) are the components of a unit vector in the hypersurface.

The resolved part of the derived vector \( N_{a;ie} \) in the direction of another unit vector \( a \) in the hypersurface is given by
\[ (- \epsilon^i \Omega_{ij} g^{ik} y_s) a_{i;k} = - \epsilon^i \Omega_{ij} g^{ik} y_s a^l, \]
(2.9)
and therefore the resolved part of the derived vector \( N_{a;ie} \) along \( \epsilon^i \) is 
\[ e^i = - \Omega_{ij} e^j. \]  
(2.10)

3. Let \( \lambda^a \) be the components of a unit vector in the direction of a curve of a congruence such that one curve of the congruence passes through each point of \( V_n \).

\( \lambda^a \) can be expressed in terms of \( y^a_{;i} \) and \( N^a \) as
\[ \lambda^a = y^a_{;i} t^i + r N^a. \]  
(3.1)
Since \( \lambda^a \) are the components of a unit vector,
\[ 1 = a_{a\beta} \lambda^a \lambda^\delta = a_{a\beta} (y^a_{;i} t^i + r N^a) (y^\beta_{;j} t^j + r N^\beta) \]
\[ = g_{ij} t^i t^j + r^2. \]  
(3.2)
The tensor derivative of (3.1) with respect to \( x^i \) yields
\[ \lambda^a_{;j} = y^a_{;ij} t^i + y^a_{;i} t^i_{;j} + r N^a_{;j} + r N^a_{;j} \]
\[ = N^a [\Omega_{ij} t^j + r_{;j}] + y^a_{;k} [t^k_{;j} - r \Omega_{ij} g^{jk}], \]  
(3.3)
From (3.3) we have
\[ a_{a\beta} \left( y^\beta_{;i} \frac{dx^i}{ds} \right) \left( \lambda^a_{;j} \frac{dx^j}{ds} \right) \]
\[ = - [r \Omega_{ij} - t_{i;j}] \frac{dx^i}{ds} \frac{dx^j}{ds}. \]  
(3.4)

If the congruence be one of normals to the hypersurface, then \( t^i = 0 \), \( r = 1 \), and the right-hand member of (3.4) reduces to (2.10). The expression
\[ (r \Omega_{ij} - t_{i;j}) \frac{dx^i}{ds} \frac{dx^j}{ds} \]