Effective field $\Phi$ inside a heavy atom has been determined by Pade' approximation technique and physically interpreted.

**Introduction**

In the context of a variety of problems in many important branches of physical sciences, such as physics, astronomy, astrophysics and fluid mechanics involving either ordinary (linear or non-linear) or partial differential equations, one generally attempts to have preferably closed form solution, and not less importantly, the numerical solution in order to describe the physical system adequately. To fulfil this aim, in some cases, partial differential equations are converted to ordinary differential equations or the order of differential equations is reduced. Exact solutions are, however, rare. In view of this, it would be a great advantage if we could employ a method which may yield the solution directly or which may be computationally efficient and economical. As we will now show in the present problem, one such method is known as Pade' approximation technique. As in the author's recent work [1], this method helps to obtain the analytical solution directly (without the need of reducing the order of equation) and is applicable to both linear and non-linear forms of the differential equations (as in case of relativistic spherical polytropes [2]). In view of its wide applications in astrophysical problems, fluid mechanics, etc., (Seidov, Sharma and Kuzakhmedov [3]; Seidov [4]; Sharma [2]; Scharlemann and Wagoner [5]; Tassoul, [6]), Pade' (2, 2) approximant method has been considered in the present problem to determine the effective field $\Phi$ inside a heavy atom. This might provide a fruitful illustration to the research workers, physicists or astrophysicists who might possibly be engaged in resolving complicated types of differential equations.

In heavy atoms there persists an effective field given by the potential $V(r)$ ($r$ denotes the distance from the nucleus) which can be self-consistently determined by the nuclear charge and distribution of electrons. To calculate the classical field $V(r)$ or $\Phi(x)$
what is needed is to solve the dimensionless Thomas—Fermi equation

$$\frac{1}{x^2} \frac{d^2 \Phi}{dx^2} = \Phi^3,$$  \hspace{1cm} (1)

as obtained by combining

$$\bar{\rho}(r) = \frac{8\pi}{3h^3} (2m)^{\frac{3}{2}} [E_f - V(r)]^{\frac{3}{2}},$$  \hspace{1cm} (2)

with the Poisson’s equation

$$\nabla^2 V = -4\pi e^2 \bar{\rho}(r),$$  \hspace{1cm} (3)

where

$$V(r) - E_s = -\frac{ze^2}{r} \Phi; \quad r = bx; \quad b = \frac{0.8853a_0}{z^3}.\hspace{1cm} (4)$$

Equation (1) satisfies the initial conditions

$$\Phi(0) = 1, \quad \Phi'(0) = -1.60. \hspace{1cm} (5)$$

$E_f, \bar{\rho}(r)$ and $a_0$ are, respectively, the total energy, density of an electron and the first Bahr radius for hydrogen. Other symbols have their usual meanings. Some efforts (Miranda, [7]; Gombás, [8]) have been made to calculate the classical field for small $x$ (for large $x$ the exact solution is known) by the numerical method.

No analytical solution of Eq. (1) for small but finite $x$ has so far been reported. Therefore in this short communication, we present analytical results by an effective and powerful method known as Pade’ $(2, 2)$ approximation technique (Baker, [9]). Results of our calculations are presented in Fig. 1 and compared with previously known values.

**Method of solutions of Eq. (1) and physical interpretation**

For small $x$, it is possible to construct the series expansion of $\Phi(x)$ of the form

$$\Phi = \sum_{i=0}^{l} a_i x^i \quad (i = 0, 2, 3, 4, 5, 6; \quad a_0 = 1). \hspace{1cm} (6)$$

This will generate the various solutions as the initial slope $\Phi'(0) = a_2(= -1.60, -1.589,$ for example) is varied. The values of the remaining coefficients are given by

$$a_3 = \frac{4}{3}, \quad a_4 = 0, \quad a_5 = (2/5)a_2, \quad a_6 = 1/3. \hspace{1cm} (7)$$

The Pade’ $(2, 2)$ approximant to the function $\Phi$ is the function given in rational form:

$$\Phi = \frac{1 + A\eta + B\eta^2 + C\eta^3}{1 + D\eta + E\eta^2 + F\eta^3}, \quad \eta = \frac{1}{x^2}, \hspace{1cm} (8)$$

where

$$D = a_2(a_3a_5 - a_2a_6)/\Delta, \quad E = a_3(a_3a_5 - a_2a_6)/\Delta,$$

$$F = (a_2a_5^2 - a_6a_3^2)/\Delta, \quad \Delta = (a_3^2 - a_5a_6^2).$$

*Acta Physica Hungarica* 59, 1986