SPHERICAL FUNCTIONS OF THE LORENTZ GROUP 
ON THE HYPERBOLOIDS

M. HUŠZÁR

Central Research Institute for Physics
1525 Budapest, Hungary

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By means of suitable coordinate systems spherical functions of the Lorentz group are derived 
on the double-sheeted, single-sheeted hyperboloids and similarly on the light cone. These form 
complete orthonormal sets of functions on each hyperboloid as well as on the light cone and 
transform according to the principal series of unitary representations of the Lorentz group. The 
spherical functions are given in a basis defined by the two-dimensional momentum \( (N_1 + M_2, N_2 - M_1) \) of horospheric translations where \( N \) and \( M \) denote the generators of Lorentz transformations and spatial rotations. It is known that the upper sheet of the double-sheeted hyperboloid is a 
homogeneous space even under the subgroup \( \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \). As a consequence of this the 
spherical functions have rather simple transformation properties under the Lorentz group.

It was shown a long time ago that matrices of unitary representations of 
the Lorentz group in horospheric basis take a form simpler than in any other basis 
provided a suitable parametrization of the group is chosen [1]. The horospheric basis 
is defined by the eigenvalues of the generators \( P_1 = N_1 + M_2 , P_2 = N_2 - M_1 \), where \( M = (M_1, M_2, M_3) \) and \( N = (N_1, N_2, N_3) \) denote the generators of spatial rotations and boosts, respectively. These are infinitesimal generators of the horospheric subgroup 
\( \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \) isomorphic to the Euclidean translation group in two dimensions. 
Quantities \( P_1, P_2 \) are the momenta corresponding to the horospheric translations. 
Physical meaning of the momenta defined so is that \( P_1/p^-, P_2/p^- \) (with the ordinary 
momentum \( p^- = p^0 - p^3 \)) can be interpreted as the components of the impact 
parameter [1].

In [1] a suitable parametrization of the \( \text{SL}(2, \mathbb{C}) \) group has been defined. As 
mentioned there the \( \gamma = 0 \) plane is a singular surface of the parametrization of 
\( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \) provided the above parametrization is used. The matrices 
\( \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \) 
can be achieved only by a limit process. Note that most global parametrizations of Lie 
groups have such a singular surface which, however, does not raise difficulties due to its 
lower dimensionality, i.e. zero measure.

A further property of the above subgroup is that on certain conditions the Lorentz 
group contracts into this when boosted to a frame moving with the velocity of light.
In what follows the subgroup \( \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \) is considered for real \( \alpha > 0 \) values and it is shown that just this subgroup, singular for the SL(2, C) representations, is of some importance for spherical functions on the hyperboloids. It is known that the hyperboloid \( H^+_+ \) \((p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = m^2 > 0, p^0 > 0)\) is a space homogeneous not only under SL(2, C) but under the subgroup \( h = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \) too. Moreover, there is a one-to-one correspondence between \( h \) and the points of \( H^+_+ \).

It is a consequence of this that spherical functions of the Lorentz group can be parametrized by the subgroup \( h \), \( Y^\sigma(p) = Y^\sigma(\alpha, \beta) \) where \( \sigma \) labels an irreducible representation and \( P = (P_1, P_2) \) is the horospheric momentum. Spherical functions transform according to irreducible representations of the Lorentz group \( g \),

\[
T_g Y^\sigma(p) = Y^\sigma(g^{-1}p) = \int d^2QD^\sigma_{QP}(g) Y^\sigma_Q(p).
\]

However, as mentioned above, to each \( g \in \text{SL}(2, C) \) an \( h = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \) can be found which has the same action on \( p \) as \( g \), i.e. \( g^{-1}p = h^{-1}p \), therefore, the matrix elements \( D^\sigma_{QP}(h) \) are sufficient for determining the transformation properties of spherical functions. The point is, however, that in this case the representation reduces to a \( \delta \)-function which implies a rather simple transformation of the spherical function,

\[
Y^\sigma(h^{-1}p) = \int d^2QD^\sigma_{QP}(h) Y^\sigma_Q(p) = \alpha^{-2} e^{i\alpha^{-1}P\beta} Y^\sigma_{\alpha^{-1}P}(p)
\]

with

\[
h = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \alpha > 0, \quad \beta = \beta_1 + i\beta_2, \quad P\beta = P_1\beta_1 + P_2\beta_2.
\]

Actually, this is an addition theorem for spherical functions.

The aim of the present paper is to show that the horospheric basis is of some importance as the spherical functions in this basis have transformation properties simpler than in any other basis considered so far. (Spherical functions of the Lorentz group in several bases can be found in [2], however, this basis is not considered there.)

The situation is similar on the light cone but it is somewhat different on the single-sheeted hyperboloid \( H^- \), \( q^2 = -M^2 < 0 \) which is not a homogeneous space under the subgroup \( h \) but rather splits into two homogeneous spaces according to whether \( q^0 - q^3 > 0 \) or \( q^0 - q^3 < 0 \). The plane \( q^0 - q^3 = 0 \) is a singular surface of the parametrization.

In Section 1 the simultaneous eigenvalue equations of the Casimir operator and those of horospheric momenta are solved on the hyperboloid \( H^+_+ \). Equations for the horospheric momentum imply a two-dimensional plane wave factor common for each hyperboloid and cone. Eigenfunctions are expressed in terms of the Bessel function \( K_{i\sigma} \). As a complete set of functions this was found a long time ago [3]. The spherical functions on the cone are expressed by a simple homogeneous function. The situation is somewhat more complicated on the single-sheeted hyperboloid where the Casimir