ON THE INITIAL-VALUE PROBLEM
OF LINEARIZED EINSTEIN’S EQUATIONS

A. MÉSZÁROS

Central Research Institute for Physics
1525 Budapest, Hungary*

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It is shown that in the absence of sourceless weak gravitational waves the linearized theory of gravitation is necessarily Lorentz covariant.

It is well-known (see, e.g. [1], Chapter 10) that the investigation of weak (linearized) Einsteinian gravitation is usually done in harmonic systems only. The form of linearized Einstein’s equations in an arbitrarily chosen \((x^0, x^1, x^2, x^3)\) harmonic system is

\[
\square h^{ij}(x) = 16\pi G \left( T^{ij}(x) - \frac{1}{2} \eta^{ij} : T(x) \right), \quad T(x) \equiv T^i_i(x),
\]

\[
h^{ij}(x), \quad j = \frac{1}{2} h(x)^i_i, \quad h(x) \equiv h_i^i(x).
\]

In this paper the dependence on \(x\) denotes the dependence on four \(x^i\) coordinates; \(T^{ij}(x)\) is the energy-momentum tensor; the index after a comma denotes partial differentiation (e.g. \(h^{ij}(x) = \frac{\partial h^{ij}(x)}{\partial x^l} \)); the system \(c = G = 1\) is used, \(c\) is the velocity of light and \(G\) is the gravitational constant; \(\eta^{ij} = \eta_{ij} \equiv \text{diag} (1, -1, -1, -1)\) is the Minkowski’s tensor; the indices are moved using the Minkowski’s tensor; \(\square \equiv \frac{\partial^2}{\partial x^i \partial x^i}\). The ten \(h^{ij}(x)\)’s are defined by

\[
g^{ij}(x) = \eta^{ij} + h^{ij}(x), \quad |h^{ij}(x)| \ll 1,
\]

where \(g^{ij}(x)\) is the contravariant metric tensor. If \((x) \equiv (x^0, x^1, x^2, x^3)\) is a harmonic system, then the transformed system \((\tilde{x}) \equiv (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)\) is a harmonic one too, if the form of the coordinate transformation is given by

\[
\tilde{x}^i(x) = x^i + y^i(x), \quad \square y^i(x) = 0,
\]

* Permanent address: Department of Astronomy and Astrophysics, Charles University, 15000 Prague-5, Švédska 8, ČSSR
where \( y^i(x)'s \) are infinitesimal. The relations (1)-(4) in detail are given, e.g. in [1] in Chapters 7.4 and 10.1.

In linearized Einstein's theory the global Lorentz transformations

\[
\tilde{x}^i(x) = L_j^i \cdot x^j + b^i
\]

(5)

are allowed too (see, e.g., [2], Box. 18.2), where \( L_j^i, b^i \) are constants. Therefore we shall assume that in (4) \( y^i(x) \) may be given by

\[
y^i(x) = L_j^i \cdot x^j - x^i + b^i,
\]

(6)

too.

The solutions of wave equations (1) are usually given by the well-known retarded potentials (see, e.g. [2], Chapter 18.4). This paper investigates again the rightfulness of these solutions.

From the theory of wave equation (see, e.g. [3]) it follows that equations (1) have unambiguous solutions if \( h^{ij}(x) \) and \( h^{ij}(x),_k \) are given on a spacelike hypersurface, which is defined by relation \( p(x) = 0 \) and is extending to spacelike infinity. In other words the initial-value data \( h^{ij}(x)|_{p(x)=0} \) and \( h^{ij}(x),_k|_{p(x)=0} \) are given. In this case the wave equations (1) have unambiguous solutions “above” the \( p(x) = 0 \) hypersurface. (An \((x^0, x^1, x^2, x^3)\) point is “above” the \( p(x) = 0 \) if \( x^0 > x^*0 \), where \( x^*0 \) is defined by \( p(x^*0, x^1, x^2, x^3) = 0 \). The set of all points “above” gives the “above” part of space-time, i.e. the future Cauchy development of \( p(x) = 0 \); see, e.g. [4], Chapter 6.5. We many solve the Eq. (1) in the past Cauchy development of \( p(x) = 0 \), too.) The hypersurface \( p(x) = 0 \) is usually defined by \( p(x) \equiv x^0 - \text{const} = 0 \); in this case this hypersurface is the entire three-dimensional space at the time \( x^0 = \text{const} \).

Let the \( h^{ij}(x)|_{p(x)=0} \) and \( h^{ij}(x),_k|_{p(x)=0} \) initial-value data be given, then the solutions of (1) “above” \( p(x) = 0 \) are given by

\[
h^{ij}(x) = h^{ij}(x) + h^{ij}(x).
\]

(7)

The ten \( h^{ij}(x)'s \) are the solutions of (1) if \( h^{ij}(x)|_{p(x)=0} = 0 \) and \( h^{ij}(x),_k|_{p(x)=0} = 0 \) (these are the so-called zero initial-value data); the ten \( h^{ij}(x)'s \) are the solutions of (1) with \( T^{ij}(x) - \frac{1}{2} \eta^{ij} \cdot T(x) = 0 \), but \( h^{ij}(x)|_{p(x)=0} = 0 \) and \( h^{ij}(x),_k|_{p(x)=0} = 0 \) are non-zero. This decomposition of the solution of the wave equation is explained in [3], Chapter 33. The ten \( h^{ij}(x)'s \) are zero if \( T^{ij}(x) - \frac{1}{2} \eta^{ij} \cdot T(x) = 0 \); the ten \( h^{ij}(x)'s \) are zero if the initial-value data are zero. The ten \( h^{ij}(x)'s \) are the solutions with sources (“the weak gravitational waves having sources”); the ten \( h^{ij}(x)'s \) are the solutions having no sources (“the sourceless weak gravitational waves”). From the theory of wave equation ([3], Chapter 34.5) it follows that the ten \( h^{ij}(x)'s \) are the retarded potentials if \( p(x) = x^0 - \text{const} = 0 \). Therefore the zero initial-value data are the necessary conditions (1) to have solutions containing retarded potentials only. If \( p(x) \equiv x^0 - \text{const} = 0 \) is chosen, then the zero initial-value data are sufficient conditions too.