ON A PROBLEM OF SPONTANEOUS COMPACTIFICATION*

P. Forgács**
Department of Physics, The University, Southampton
SO9 5NH England

Z. Horváth and L. Palla
Institute for Theoretical Physics, Roland Eötvös University
1088 Budapest, Hungary

We investigate how the equations governing spontaneous compactification to non-symmetric coset spaces can be solved for two particular classes of internal spaces.

1. Introduction

In the last few years we witnessed a revival of interest in using theories based on the old assumption that the physical space-time has more than four dimensions to explain the observed properties of elementary particles. The spontaneously compactified theories [1] form a class of these models: they are theories of gravity coupled to Yang–Mills and spinor fields in a $4 + d$ dimensional space-time which are reduced — via a solution of the Einstein–Yang–Mills (EYM) field equations — to a 4 dimensional low energy theory by compactifying the extra $d$ dimensions into a compact coset space $S/H$ with an appropriately small size $R_0$. Recent investigations showed that it is rather difficult to find a model explaining all details of unified gauge theories [2]. In these models mostly symmetric coset spaces were used to describe the extra compact dimensions [3]. Quite recently the equations governing spontaneous compactification to non-symmetric coset spaces (NSCS) have been derived [4, 5] and it was shown [4] that in contrast to the case of symmetric spaces they cannot be solved for all NSCS. The aim of this paper is to determine a class of NSCS for which the compactification equations do admit a solution.

In the next Section we briefly outline the derivation of the equations governing spontaneous compactification to NSCS while in Section 3 we show how these equations can be solved in two special cases, namely for $S/H$'s having the form of

$$\text{SU}(k + l)/\text{SU}(k) \times U_1 \times \ldots \times U_1$$

or

$$\text{SO}(2[k + l - 1])/\text{SU}(k) \times U_1 \times \ldots \times U_1$$

where $k$ and $l$ are arbitrary integers ($k \geq 1$, $l \geq 2$).

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** On leave from Central Research Institute for Physics, H-1525 Budapest 114, Hungary
2. Spontaneous compactification to NSCS

We start with the standard EYM action for a coupled system of gravity and
gauge fields with gauge group $G_{YM}$ in $4+d$ dimensions

$$S = \int dz^{4+d}(-g)^{1/2}\left(\frac{R}{\kappa^2} - \frac{1}{4g^2}F_{AB}^aF^{aAB}_A - \Lambda\right),$$

where $\kappa$ and $g$ are the $4+d$-dimensional gravitational and gauge coupling constants
and $\Lambda$ is a cosmological constant. The equations following from (1) are $4+d$
dimensional Einstein equations and the Yang–Mills equations coupled to $4+d$
dimensional gravity. Spontaneous compactification means that we find a solution of
this coupled system of equations where the $4+d$ dimensional space-time is the direct
product of the flat 4 dimensional Minkowski space ($M_4$) and a compact coset space $S/H$
(dim $S$—dim $H = d$). To obtain $M_4 \times S/H$ as a solution we must have non-vanishing
gauge fields on the internal space $S/H$, and as this internal space admits $S$ as a symmetry
group we also require that the explicit gauge fields be $S$ symmetric in the sense of [6]. In
this paper we assume that $S$ is a simple Lie group and $S/H$ is a NSCS.

To describe $M_4 \times S/H$ we split the coordinates $z^M = (x^m, y^\#)$ $m = 0, 1, \ldots, 3$; $\# = 1, 2, \ldots d$ denoting Minkowski versus internal ones. Furthermore, we choose from each
coset an element labelled by $y^\#$ denoted as $L(y)$. Using this quantity the vielbeins $e^a$
on $S/H$ are given by [7]

$$L^{-1}(y)dx^m = e^aQ_a + \omega^iQ_i = (e^a_m(y)Q_a + \omega^i_m(y)Q_i)dy^\#.$$  (2)

Here $Q_A$, $A = 1, \ldots, \dim S$, are the generators of $S$ satisfying $[Q_A, Q_B] = f_{ABC}Q_C$ with
$f_{ABC}$ being the completely antisymmetric structure constants of $S$. Furthermore, we
denote the indices corresponding to $H$ and to the coset $S/H$ by $i$ and $a$, respectively (so $A$
$= (i, a)) As $H$ is a closed subgroup of $S$, $f_{ijk} = 0$ and $f_{ijk}$ are the structure constants of $H$. The fact that $S/H$ is not a symmetric space means that $f_{abc} \neq 0$. Since $e^a$ and $\omega^i$ satisfy the
Maurer–Cartan equations — as a consequence of Eq. (2) — we determine the torsion
free (“natural”) connection on $S/H$ as

$$B^a_b = -\frac{1}{2} f^a_{bc}e^c - f^a_{bi}\omega^i.$$  (3)

It is straightforward to determine the Riemann and Ricci tensors for this
connection, in fact for the latter we obtain

$$R_{ab} = \frac{1}{4} (f_{bec}f_{eca} + 4f_{cai}f_{icb}.$$  (4)

To make an ansatz for the compactifying gauge fields we must choose appropriate $G_{YM}$
admitting the possibility of constructing $S$ symmetric gauge configurations. The