ANGULAR MOMENTUM AND UNITARY SPINOR BASES OF THE LORENTZ GROUP*

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The matrix of the boost operator has a rather complicated form in angular momentum basis while it reduces to a diagonal form $e^{i\sigma \tau}$ in the unitary spinor basis. The overlap coefficients between the two bases turn out to be complex Wigner coefficients of the rotation group. With the aid of these coefficients an expression for the boost function is derived in terms of a power series in $e^{-2\alpha}$.

It has been shown in [1] and in the revised versions [2, 3] that if the notion of spinor is generalized to the unitary case, the unitary representations of the Lorentz group can be expressed in a simpler form than those in O(3), O(2, 1), E(2) bases. In the spinor basis the Lorentz group figures as an SO(3, C) group (the group of three-dimensional complex rotations) isomorphic with the proper Lorentz group. Such an interpretation is possible since there exists a combination of the generators satisfying the commutators of two independent angular momenta. A similar decomposition holds for the O(4) group. Though in the above spinor basis Lie Algebras of the O(3, 1) and the O(4) groups coincide, for unitary representations all generators of the latter group are Hermitean, whereas in the Lorentz group the generators of the two complex angular momenta are pairwise adjoint to each other. In the spinor basis the unitary representations of the O(4) group are simply a product of two D-functions and the transition to the angular momentum basis can be accomplished by means of the Wigner coefficients of the real rotation group. Hence it seems natural that the overlap coefficients between O(3) and O(2, C) (spinor) bases of the Lorentz group can be interpreted as an analytic continuation of the familiar Wigner coefficients [2]. The overlap coefficients have been derived earlier for a special case by KUZNETSOV et al. [4].

The expression for the boost function $d_{\mu \nu}^\sigma (\alpha)$ in the angular momentum basis is rather complicated. On the other hand, in the spinor basis the matrix of the boost operator along the z-axis is diagonal and has the form $e^{i\alpha \tau}$. The purpose of this paper is to show that complicated formulas containing multiple sums for the function $d_{\mu \nu}^\sigma (\alpha)$ can be reduced to the product of two overlap coefficients.

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which accomplish the transition between \(O(3)\) and \(O(2, \mathbb{C})\) bases. These coefficients are essentially complex Wigner coefficients and can be treated easily since they possess a number of properties well-known from the rotation group.

In Section 1 the eigenfunctions of the \(O(3)\) and \(O(2, \mathbb{C})\) bases are constructed. In Section 2 the overlap coefficients between the two bases are calculated and in Section 3 it is shown that they are essentially complexified Wigner coefficients of the real rotation group. In Section 4 an explicit form for the boost function is presented in terms of a power series of the form \(\sum_k a_k e^{-2ka}\).

1. The basis functions

Denoting the infinitesimal generators of spatial rotations and boosts along the \(k\)-axis \((k = 1, 2, 3)\) by \(M_k\) and \(N_k\), respectively, the combinations

\[
\frac{1}{2} (M_k + i N_k) = J_k \quad \text{and} \quad \frac{1}{2} (M_k - i N_k) = K_k
\]

satisfy

\[
[J_k, J_l] = i \varepsilon_{klm} J_m, \quad [K_k, K_l] = i \varepsilon_{klm} K_m, \quad [J_k, K_l] = 0.
\]

For unitary representations \(J_k = K_k^+\) holds. The Casimir operators \(J^2, K^2\) of the two complex angular momenta \(J, K\) are the Casimir operators of the Lorentz group:

\[
J^2 | > = j(j+1) | >, \quad K^2 | > = j^*(j^*+1) | >,
\]

where

\[
j = \frac{1}{2} (j_0 - 1 + i\sigma),
\]

\((j_0 = 0, \pm \frac{1}{2}, \pm 1, \ldots, 0 \leq \sigma < \infty \text{ for the principal series})\). Irreducible unitary representations can be characterized by \((j_0, \sigma)\) or by \((j, j^*)\).

In unitary spinor basis the generators \(M_3\) and \(N_3\) are diagonal:

\[
M_3 | j j^*; \mu \nu > = \mu | j j^*; \mu \nu >, \quad N_3 | j j^*; \mu \nu > = v | j j^*; \mu \nu >,
\]

where \(\mu\) takes integer (half-integer) values for single-(double-) valued representations.

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