A Convergence Theorem for Newton's Method in Banach Spaces*

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Received October 14, 1985

On the basis of the results obtained in a series of papers [25]-[28], a convergence theorem for Newton's method in Banach spaces is given, which improves the theorems of Kantorovich [4], Lancaster [8] and Ostrowski [10]. The error bounds obtained improve the recent results of Potra [17].

Key words: convergence theorem, Newton's method, Kantorovich theorem, Lancaster's theorem, Ostrowski's theorem, error estimates, Potra's bounds

1. Introduction

There is much literature concerning convergence and error estimates for Newton's method in Banach spaces. In a series of papers [25]-[28], we examined the error bounds which have been obtained by many authors (Dennis [1], Tapia [24], Rall-Tapia [20], Ostrowski [15], [16], Gragg-Tapia [3], Miel [9]-[11], Potra-Ptáč [18], Moret [12]) under assumptions of the Kantorovich theorem, and compared them with the Kantorovich bounds. We concluded [28] that their results follow from the Kantorovich theorem so that, under the Kantorovich assumptions, the Kantorovich theorem still gives us the best upper bounds for the Newton method.

In this paper, we are interested in improving the assumptions of the Kantorovich theorem and the assertions of the Ostrowski theorem [16; Theorem 38.1]. We shall first state both theorems and several lemmas in §2. Next, in §3, we shall present a convergence theorem which improves both theorems. It will also be shown that results improve the error bounds of Lancaster [8], Kornstaedt [7] and Potra [17]. Finally, in §4, we shall show that Ostrowski's other theorem [16; Theorem 38.2] can be derived by our approach.

2. Preliminaries

Let $X$ and $Y$ be Banach spaces and consider an operator $F: D \subseteq X \rightarrow Y$. If $F$ is Fréchet differentiable in an open convex set $D_0 \subseteq D$, then the Newton method for solving the equation

* Sponsored by the United States Army under Contract No. DAAG29-80-C-0041, and by the Ministry of Education in Japan.
\[ F(x) = 0 \] 

is defined by 

\[ x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n \geq 0, \tag{2.2} \]

provided that \( F'(x_n)^{-1} \in L(Y, X) \) exists at each step, where \( L(Y, X) \) denotes the Banach space of bounded linear operators of \( Y \) into \( X \). Sufficient conditions for convergence of the iterates (2.2), error estimates and existence and uniqueness regions of solutions are given by the famous Kantorovich theorem:

**Theorem 2.1** (Kantorovich [4], [5] and Kantorovich-Akilov [6]). Let \( F: D \subseteq X \rightarrow Y \) be Fréchet differentiable in an open convex set \( D_0 \subseteq D \). Assume that for some \( x_0 \in D_0 \), \( F'(x_0) \) is invertible and that

\[
\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq K\|x - y\|, \quad K > 0, \quad x, y \in D_0,
\]

\[
\|F'(x_0)^{-1}F(x_0)\| \leq \eta, \quad \eta > 0,
\]

\[ h = K\eta \leq \frac{1}{2} \]

and

\[
S(x_0, t^*) = \left\{ x \in X \left| \|x - x_0\| \leq t^* = \frac{1 - \sqrt{1 - 2h}}{K} \right. \right\} \subseteq D_0.
\]

Then:

(i) The iterates (2.2) are well-defined, lie in the open ball \( S(x_0, t^*) = \{x \in X \mid \|x - x_0\| < t^*\} \) and converge to a solution \( x^* \) of the equation (2.1).

(ii) The solution \( x^* \) is unique in \( S(x_0, t^{**}) \cap D_0 \) if \( 2h < 1 \) and in \( S(x_0, t^{**}) \) if \( 2h = 1 \), where \( t^{**} = (1 + \sqrt{1 - 2h})/K \).

(iii) Error estimates

\[
\|x^* - x_n\| \leq \frac{2\eta_n}{1 + \sqrt{1 - 2h_n}} \leq 2^{1-n}(2h)^{2n-1}\eta, \quad n \geq 0, \tag{2.3}
\]

hold, where \( \eta_n \) and \( h_n \) are defined by the recurrence relations

\[
B_0 = 1, \quad \eta_0 = \eta, \quad h_0 = h = K\eta,
\]

\[ B_n = \frac{B_{n-1}}{1 - h_{n-1}}, \quad \eta_n = \frac{h_{n-1}\eta_{n-1}}{2(1 - h_{n-1})}, \quad h_n = KB_n\eta_n, \quad n \geq 1. \tag{2.4}
\]

(iii)' Put \( f(t) = \frac{1}{2}Kt^2 - t + \eta \) and define the sequence \( \{t_n\} \) by

\[ t_0 = 0, \quad t_{n+1} = t_n - f(t_n)/f'(t_n), \quad n \geq 0. \]

Then

\[ \|x_{n+1} - x_n\| \leq t_{n+1} - t_n. \]