Limit of Julia sets for \( z^d + c \)

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Abstract The limit behavior of Julia set \( J(f_{d,c}) \) for polynomials \( f_{d,c}(z) = z^d + c \) is considered. That \( \{ J(f_{d,c}) \}_{d \geq 2} \) converges to the unit circle \( S^1 \) in Hausdorff metric for some fixed parameter \( c \) is proved and some examples showing \( \{ J(f_{d,c}) \}_{d \geq 2} \) has no limit are given.

Keywords: Julia set, capacity, Hausdorff metric.

The dynamics of polynomials \( f_{d,c}(z) = z^d + c \) has been studied by several authors\(^{[1,2]}\). In this note, we will consider how \( J(f_{d,c}) \) varies for fixed \( c \) as \( d \) tending to infinity.

For fixed \( c \), we establish several lemmas at first.

Lemma 1. For any positive \( \varepsilon \), there exists \( d_0 \) such that the Julia set \( J(f_{d,c}) \) is contained in the closed disk \( \{ |z| < 1 + \varepsilon \} \) for \( d \geq d_0 \).

Proof. For any positive \( \varepsilon \), there exists \( d_0 \) such that \( (1 + \varepsilon)^d - |c| > 1 + \varepsilon \) for \( d \geq d_0 \). If \( |z| > 1 + \varepsilon , |f_{d,c}^n(z)| > 1 + \varepsilon \) for all \( n > 0 \). The Montel’s theorem implies that \( \{ f_{d,c}^n \} \) is a normal family in \( \{ |z| > 1 + \varepsilon \} \). \( J(f_{d,c}) \) is contained in \( \{ |z| < 1 + \varepsilon \} \) for \( d \geq d_0 \).

Lemma 2. If \( |c| \neq 1 \), for any positive \( \varepsilon \), there exists \( d_0 \) such that \( J(f_{d,c}) \) is contained in \( \{ |z| > 1 - \varepsilon \} \) for \( d \geq d_0 \).

Proof. For any positive \( \varepsilon \), there exists \( d_0 \) such that \( (1 - \varepsilon)^d < \frac{\delta}{2} \) and \( (1 + \frac{\delta}{2})^d > 2 + \frac{3\delta}{2} \) for \( d \geq d_0 \).

If \( |z| < 1 - \varepsilon , |f_{d,c}^n(z)| > 1 + \frac{\delta}{2} \) for all \( n \geq 2 \). The family \( \{ f_{d,c}^n \} \) is normal in \( \{ |z| < 1 - \varepsilon \} \) and \( J(f_{d,c}) \) is contained in \( \{ |z| > 1 - \varepsilon \} \) for \( d \geq d_0 \).

Lemma 3. Let \( E \) be a bounded closed set on \( \mathbb{C} \) and \( E \) be not a single point set. Then \( d(E) \geq 2 \cdot \text{cap}(E) \) and equality holds if and only if the outer boundary \( \Gamma \) of \( E \) is a circle, where \( d(E) \) is the diameter of \( E \), \( \text{cap}(E) \) is the logarithmic capacity of \( E \), \( \Gamma = \partial D_\infty \), \( D_\infty \) is the unbounded component of \( \mathbb{C} - E \).

Proof. Set
\[
E_1 = \bigcup_{\varepsilon > 0} \left\{ \sum_{i=1}^{n_1} a_i x_i \mid 0 \leq a_i \leq 1, \sum_{i=1}^{n_1} a_i = 1 \right\},
\]
where \( E_1 \) is the closure of \( E_1 \), then \( E_1 \) is a convex set, \( d(E_1) = d(\bar{E}_1) \) and \( \text{cap}(E_1) \geq \text{cap}(E) \).

If \( \text{cap}(E) = 0 \), then statement is true.

We suppose \( \text{cap}(E) > 0 \).

Let \( E_2 = E_1 / \text{cap}(E_1) \). Then \( \text{cap}(E_2) = 1 \) and \( d(E_2) = d(\bar{E}_1) / \text{cap}(E_1) \). There exists a conformal mapping \( h(z) = z + z_0 + \frac{a_1}{2} + \cdots \) from \( \mathbb{C} - E_2 \) onto \( \{ |z| > R \} \). By a theorem of ref. \([4]\), \( R = \text{cap}(E_2) = 1 \). Hence \( d(E_2) \geq 2 \) and \( d(E_2) = 2 \) if and only if \( \partial E_2 \) is a circle\(^{[5]} \). Therefore, \( d(E) = \)
NOTES

d(\overline{E}_1) \geq 2 \cdot \text{cap}(\overline{E}_1) \geq 2 \cdot \text{cap}(E).

If the outer boundary \Gamma of \mathcal{E} is a circle \{|z| = r\}, then d(E) = 2 \cdot \text{cap}(E) = 2r.

If d(E) = 2 \cdot \text{cap}(E), then d(\overline{E}_1) = 2 \cdot \text{cap}(\overline{E}_1). \partial \overline{E}_1 is a circle. It suffices to prove \partial \overline{E}_1 \subset \mathcal{E}. If it is false, there exists a point \(z_0 \in \partial \overline{E}_1\) and \(z_0 \notin \mathcal{E}\). Since \mathcal{E} is a compact set on \mathbb{C}, we choose an open set \(U(z_0)\) containing \(z_0\) such that \(U(z_0) \cap \mathcal{E} = \emptyset\). Let \(F = \overline{E}_1 \setminus U(z_0) \supset \mathcal{E}\). Then \(d(E) = d(\overline{E}_1) \geq d(F) \geq 2 \cdot \text{cap}(F) \geq 2 \cdot \text{cap}(E)\). Therefore \(d(E) = 2 \cdot \text{cap}(E)\) implies \(d(F) = 2 \cdot \text{cap}(F)\), \(\partial F\) is not a circle. It is impossible. Hence \(\partial \overline{E}_1 = \mathcal{E}\). The outer boundary \(\Gamma = \partial \overline{E}_1\) of \mathcal{E} is a circle.

Q.E.D.

Lemma 4. For any point \(z\) on the unit circle \(S^1\) and any neighbourhood \(U\) of \(z\), there exists \(d_0\) such that \(J(f_{d_0}, c) \cap U \neq \emptyset\) for \(d \geq d_0\).

Proof. If it is false, there are \(z_0 \in S^1\) and a neighborhood \(U_0\) of \(z_0\) such that \(J(f_{d_0}, c) \cap U_0 = \emptyset\) for a sequence \(\{d_n\}\) tending to infinity.

Denote \(V_\epsilon = \{|z| < 1 + \epsilon\} - U_0\) and \(V_0 = \{|z| \leq 1\} - U_0\). Lemma 3 implies \(\text{cap}(V_0) < 1\). A theorem in ref. [4] says that \(\text{cap}(V_\epsilon) \rightarrow \text{cap}(V_0)\), \(\text{cap}(V_{\epsilon_0}) < 1\) for some small \(\epsilon_0\). From Lemma 1 and the above assumption, \(J(f_{d_0}, c)\) is contained in \(V_\epsilon\) for \(n\) large enough. \(\text{cap}(J(f_{d_0}, c)) \leq \text{cap}(V_{\epsilon_0}) < 1\). This contradicts with \(\text{cap}(J(f_{d_0}, c)) = 1^{[d]}\).

Q.E.D.

Lemmas 1, 2 and 4 imply the following theorem:

Theorem 1. For any fixed \(c\) (\(|c| \neq 1\)), \(J(f_{d_0}, c) \rightarrow S^1\) in Hausdorff metric as \(d \rightarrow \infty\).

For \(|c| = 1\), \(c = e^{2\pi i \theta}\), we consider the case \(\theta = \frac{p}{q}\) being a rational number in the following. In this case, \(A = \{d \theta \mod 1\}_{\theta \in \mathbb{Q}/\mathbb{Z}}\) is a finite subset of \(\mathbb{T} = \mathbb{R}/\mathbb{Z}\).

If the orbit \(|P^q_{d_0} (0)|_{n \geq 1}\) is contained in \(S^1\) for some \(d\), then

\[
(d \theta - \theta) \mod 1 = \frac{1}{3}
\]

or

\[
(d \theta - \theta) \mod 1 = \frac{2}{3}.
\]

When \((d \theta - \theta) \mod 1 = \frac{1}{3}\), there are two possibilities

\[
\left( (d \theta - \theta) \mod 1 = \frac{1}{3}, \quad d (\theta + \frac{1}{6}) \mod 1 = \frac{1}{3} \right)
\]

or

\[
\left( (d \theta - \theta) \mod 1 = \frac{1}{3}, \quad d (\theta + \frac{1}{6}) \mod 1 = \frac{2}{3} \right).
\]

It is easy to check that the last case is impossible.

When \((d \theta - \theta) \mod 1 = \frac{2}{3}\), the same things can be discussed.

We conclude that the orbit \(|P^q_{d_0} (0)|_{n \geq 1}\) is contained in \(S^1\) if and only if

\[
\left( (d \theta - \theta) \mod 1 = \frac{1}{3}, \quad d (\theta + \frac{1}{6}) \mod 1 = \frac{1}{3} \right)
\]

or