NOTES

SVD row or column symmetric matrix

ZOU Hongxing¹, WANG Dianjun², DAI Qionghai¹ & LI Yanda¹

1. Department of Automation, Tsinghua University, Beijing 100084, China;
2. Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

Abstract A new architecture for row or column symmetric matrix called extended matrix is defined, and a precise correspondence of the singular values and singular vectors between the extended matrix and its original (namely, the mother matrix) is derived. As an illustration of potential, we show that, for a class of extended matrices, the singular value decomposition using the mother matrix rather than the extended matrix per se can save the CPU time and memory without loss of numerical precision.

Keywords: singular value decomposition, extended matrix, time-frequency distribution, signal processing.

Major shortcomings of the singular value decomposition (SVD) based techniques are excessive storage requirements and computational complexity[1]. Early on, our interest in reducing computational burden of SVD was motivated by a typical kind of symmetric phenomenon: symmetry due to one-to-one correspondence between the positive frequencies and their negative counterparts. For example, in signal processing, the commonly used short-time Fourier transform (STFT) of a real-valued signal is symmetric with respect to DC component. We found that the singular values of this whole real time-frequency plane of STFT is exactly \( \sqrt{2} \) times larger than that of the half one (positive or negative frequencies). Although this kind of difference of singular values does not affect the solutions to subspace problems (where we only concern the evolution of the singular values), the improvement in computational speed of SVD using only the half time-frequency plane is salient with considerable reduction in storage requirements. Moreover, the more the number of sample points of a signal, the more pronounced the improvement. In fact, the row or column symmetry appears not only in STFT, but also in some other time-frequency representations such as Gabor transform, spectrogram, minimum mean cross-entropy combination of spectrograms, Margenau-Hill distribution, pseudo Margenau-Hill distribution, Page distribution, pseudo Page distribution and Rihaczek distribution[2,3]. If we go beyond the time-frequency representations on the hand, we will notice that our urban or indoor world contains a plethora of symmetry or periodicity, repeating rows or columns of bricks, tiles, windows, or the like abound. By analogy with the extension of signal in power spectrum estimation, we could refer to the row or column symmetric matrix as the “extended matrix”. This note is devoted to the SVD of the extended matrix, which to our best knowledge has not been documented elsewhere.

1 Definition of extended matrix

Definition 1 (row extension). Let matrix \( A \in C^{m \times n} \), and \( k \) be a positive integer. Define matrices \( R_k(A) \) and \( R'_k(A) \) respectively by

\[
R_k(A) = \begin{bmatrix}
A \\
A \\
\vdots \\
A
\end{bmatrix} \in C^{km \times n}, \quad R'_k(A) = \begin{bmatrix}
A \\
B \\
\vdots \\
B
\end{bmatrix} \in C^{km \times n},
\]

(1)

where \( A \) is referred to as the mother matrix, \( B = J_m A, \quad J_m = [J_{i,j}] \in R^{m \times m} \) is a square matrix such that

\[
J_{1,m} = J_{2,m-1} = \cdots = J_{m,1} = 1 \quad \text{and all the other elements are zeros. In fact, if } A = \begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_m
\end{bmatrix}, \text{ then } B = \begin{bmatrix}
\alpha_m \\
\vdots \\
\alpha_1
\end{bmatrix}.
\]

The matrix \( R_k(A) \) is called the first class of row \( k \)-extension of \( A \), and correspondingly, \( R'_k(A) \) is
called the second class of row \( k \)-extension of \( A \).

**Definition 2** (column extension). Let matrix \( A \in \mathbb{C}^{m \times n} \) and \( k \) be a positive integer. Define matrices \( C_k(A) \) and \( C'_k(A) \) respectively by

\[
C_k(A) = [A, A, \cdots, A] \in \mathbb{C}^{mn \times k}, \quad C'_k(A) = [A, B, \cdots, B] \in \mathbb{C}^{mn \times k},
\]

where \( B = AJ_n \). If \( A = [\beta_1, \beta_2, \cdots, \beta_m] \), then \( B = [\beta_m, \cdots, \beta_2, \beta_1] \). The matrix \( C_k(A) \) is called the first-class column \( k \)-extension of \( A \), and \( C'_k(A) \) the second-class column \( k \)-extension of \( A \).

The following results can be obtained from the two definitions immediately.

**Lemma 1.**

(i) \( \text{rank}(R_k(A)) = \text{rank}(R'_k(A)) = \text{rank}(C_k(A)) = \text{rank}(C'_k(A)) = \text{rank}(A) \);

(ii) \( R_k(A)^H = C_k(A)^H, \quad R'_k(A)^H = C'_k(A)^H, \quad C_k(A)^H = R_k(A^H), \quad C'_k(A)^H = R'_k(A^H) \);

(iii) Let \( X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}^{n \times n} \), then

\[
R_k(XA) = R_k(X)A, \quad R_k(AY) = R_k(A)Y, \quad C_k(XA) = XC_k(A), \quad C'_k(XA) = XC'_k(A).
\]

2 SVD for extended matrix

**Theorem 1** (SVD of the first class of row \( k \)-extension of \( A \)). Let \( A \in \mathbb{C}^{m \times n}, m \geq n \). \( A = UDV^H \) be the SVD of \( A \), where unitary matrices \( U \in \mathbb{C}^{m \times m} \) and \( V \in \mathbb{C}^{n \times n} \), \( D = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \), \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n) \), with \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \). Then there exists an SVD of \( R_k(A) \) such that

\[
R_k(A) = PTQ^H,
\]

where \( T = \begin{bmatrix} A \\ 0 \end{bmatrix} \in \mathbb{C}^{mn \times mn}, \quad \Delta = \text{diag}(\delta_1, \delta_2, \cdots, \delta_n) \) with \( \delta_1 \geq \delta_2 \geq \cdots \geq \delta_n \geq 0 \), such that

(i) \( \delta_i = \sqrt{k}\sigma_i, i = 1, 2, \cdots, n \);

(ii) \( P = [P_1, P_2] \), where \( P_1 = R_k \left( \frac{1}{\sqrt{k}} U \right) \), \( P_2 \in C^{m \times (k-1)n}, P_2^H P_1 = 0 \);

(iii) \( Q = V \).

**Proof.** Since \( R_k(A)^H R_k(A) = kA^H A = kVDV^H = V(k\Sigma^2)V^H \), \( k\sigma_1^2, k\sigma_2^2, \cdots, k\sigma_n^2 \) are the eigenvalues of \( R_k(A)^H R_k(A) \). From the definition of singular values, \( \sqrt{k}\sigma_1, \sqrt{k}\sigma_2, \cdots, \sqrt{k}\sigma_n \) must be the singular values of \( R_k(A) \), hence, (i) holds.

In order to get the SVD (3) of \( R_k(A) \) satisfying conditions (ii) — (iii), let \( P_1 = R_k \left( \frac{1}{\sqrt{k}} U \right) \) \( \in \mathbb{C}^{mn \times mn} \). Since

\[
P_1^H P_1 = R_k \left( \frac{1}{\sqrt{k}} U \right)^H R_k \left( \frac{1}{\sqrt{k}} U \right) = U^H U = I_m,
\]

we have a matrix \( P_2 \in C^{mn \times (k-1)n} \), such that \( P = [P_1, P_2] \) is unitary. In this case, \( P_2^H P_1 = 0 \).

Let \( Q = V \), then

\[
P^H R_k(A)Q = [P_1, P_2]^H R_k(A)V = \left[ \begin{array}{c} P_1^H \\ P_2^H \end{array} \right] R_k(A)V = \left[ \begin{array}{c} P_1^H R_k(A)V \\ P_2^H R_k(A)V \end{array} \right].
\]