On certain estimates for Marcinkiewicz integrals and extrapolation

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Abstract

We obtain $L^p$ estimates for parametric Marcinkiewicz integrals associated to polynomial mappings and with rough kernels on the unit sphere as well as on the radial direction. These estimates will allow us to use an extrapolation argument to obtain some new and improved results on Marcinkiewicz integrals. Also, such estimates provide us with a unifying approach in dealing with Marcinkiewicz integrals when the kernel function $\Omega$ belongs to the class of block spaces $B_q^{(0,\alpha)}(S^{n-1})$ as well as when $\Omega$ belongs to the class $L(\log L)^\alpha(S^{n-1})$. Our conditions on the kernels are known to be the best possible in their respective classes.

1. Introduction

Throughout this paper, let $\mathbb{R}^n$, $n \geq 2$, be the $n$-dimensional Euclidean space and $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue surface measure $d\sigma$. 

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Also, we let $\xi'$ denote $\xi/|\xi|$ for $\xi \in \mathbb{R}^n \setminus \{0\}$ and $p'$ denote the exponent conjugate to $p$, that is $1/p + 1/p' = 1$.

Let $\mathcal{P}(y) = (P_1(y), \ldots, P_m(y))$ be a polynomial mapping, where each $P_j$ is a real-valued polynomial on $\mathbb{R}^n$. To $\mathcal{P}$ we associate a parametric Marcinkiewicz integral operator $M^p_{\mathcal{P}, \Omega, h}$ defined initially for $C_0^\infty$ functions on $\mathbb{R}^n$ by

$$M^p_{\mathcal{P}, \Omega, h}(f)(x) = \left(\int_0^\infty \left| t^{-\rho} \int_{|u| \leq t} f(x - \mathcal{P}(u)) \frac{\Omega(u')}{|u|^{n-\rho}} h(|u|) du \right|^2 \frac{dt}{t}\right)^{1/2}$$

where $\rho = \sigma + i\tau$ ($\sigma, \tau \in \mathbb{R}$ with $\sigma > 0$), $h$ is a measurable function on $\mathbb{R}_+$, and $\Omega \in L^1(S^{n-1})$ and satisfies

$$\int_{S^{n-1}} \Omega(u) d\sigma(u) = 0. \quad (1.1)$$

If $m = n$ and $\mathcal{P}(y) \equiv y$, we shall simply denote $M^p_{\mathcal{P}, \Omega, h}$ by $M^p_{\Omega, h}$ and we denote $M^p_{\Omega, h}$ by $M_{\Omega}$ if $h \equiv 1$ and $\rho = 1$.

We point out that the class of operators $M^p_{\mathcal{P}, \Omega, h}$ is related to the class of homogeneous singular integral operators

$$T_{\mathcal{P}, \Omega, h}(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \mathcal{P}(u)) \frac{\Omega(u')}{|u|^{n-\rho}} h(|u|) du. \quad (1.2)$$

The class of operators defined by (1.2) belongs to the class of singular Radon transforms and it has been studied by many authors. For more information about the importance and the recent development in the study of this class of operators, we refer the readers to [18, 10, 4, 2, 15], among others. The operators $M^p_{\mathcal{P}, \Omega, h}$ defined in (1.1) have their roots in the classical Marcinkiewicz integral operator $M_{\Omega}$. In [16], E.M. Stein introduced the operator $M_{\Omega}$ and proved that if $\Omega \in \text{Lip}_\alpha(S^{n-1})$ ($0 < \alpha \leq 1$), then $M_{\Omega}$ is of type $(p, p)$ for $1 < p \leq 2$ and of weak type $(1, 1)$. Subsequently, the study of $M_{\Omega}$ and some of its extensions has attracted the attention of many authors. Readers may consult [20, 7, 1, 3, 2, 6], among a large number of references for their development and applications. Before stating some known results relevant to our current study, we need to recall and introduce some definitions.

For $\gamma > 0$, let $\Delta_\gamma(\mathbb{R}_+)$ denote the collection of all measurable functions $h : [0, \infty) \rightarrow \mathbb{C}$ satisfying

$$\|h\|_{\Delta_\gamma} = \sup_{k \in \mathbb{Z}} \left( \int_{2^k}^{2^{k+1}} |h(t)|^\gamma dt / t \right)^{1/\gamma} < \infty$$

and $\mathcal{L}_\gamma(\mathbb{R}_+)$ denote the collection of all measurable functions $h : [0, \infty) \rightarrow \mathbb{C}$ satisfying

$$L_\gamma(h) = \sup_{k \in \mathbb{Z}} \left( \int_{2^k}^{2^{k+1}} |h(t)(\log(2 + |h(t)|))^\gamma dt / t \right) < \infty.$$

Also, we let $\mathcal{N}_\gamma(\mathbb{R}_+)$ denote the class of all measurable functions $h$ on $\mathbb{R}_+$ such that

$$\mathcal{N}_\gamma(h) = \sum_{m=1}^{\infty} m^{\gamma} 2^m d_m(h) < \infty,$$